The geometry of jet bundles, with applications to the calculus of variations

Thesis

The Geometry of Jet Bundles, with Applications to the Calculus of Variations

David John Saunders, M.A.

A dissertation offered for the degree of Ph.D.
at the Open University
in the Faculty of Mathematics
1 April 1987

Abstract

This dissertation presents an exposition of the geometric theory of jet bundles, and describes some applications of this theory to the study of certain types of differential equation, principally those associated with the calculus of variations.

The detailed structure of this dissertation is as follows. Chapter 1 contains a review of the properties of bundles and their vector fields and differential forms, and also describe the theory of derivations along bundle maps. Chapter 2 contains a coherent account of the theory of jet bundles. It includes a definition of derivations of type h_* and v_* and the vertical bracket of vector-valued forms, and also describes infinite jets in a way which clarifies the manifold structure of J^∞ and the properties of smooth functions on this space. Chapter 3 contains a description of the prolongations of bundle maps and of vector fields, and also explains some properties of repeated jets. Chapter 4 contains a definition of jet fields and their associated connections, and a summary of their properties. It also relates the construction of Bäcklund transformations to jet fields along bundle maps. Chapter 5 contains a brief history of the almost tangent structure on a tangent manifold and a generalisation of this structure to vertical endomorphisms on an arbitrary jet manifold and a vertical vector-valued m-form on a first-order jet manifold. Chapter 6 contains a short summary of the modern approach to Lagrangian and Hamiltonian systems in mechanics. It also includes a construction for a Cartan form in higher-order field theories and an explanation of the relationship of integral sections of jet fields to extremals of variational problems. Finally, Chapter 7 contains a discussion of completely integrable evolution equations and their Hamiltonian structure. There is also an explanation of the Kac-Moody algebra interpretation of the Inverse Scattering Transform in terms of higher-order tangent manifolds.

Contents

Introduction

Il est bien des manières d'employer nos mathématiques à une meilleure intelligence du concret. Dans un certain nombre de cas, elles interviennent seulement comme instrument. *Le savant veut parvenir le plus rapidement possible à une comparaison numérique avec l'expérience et assurer ainsi son pouvoir sur les choses.*

Mais les mathématique assume souvent un rôle plus ambitieux. Elle se veut aussi mode de pensée *pour appréhender la réalité et ne prétend à son intelligence que lorsqu'il a été possible de construire, pour l'ensemble des phénomènes étudiés un modèle mathématique* cohérent. *Les grandes théories physiques de notre temps aboutissent à la création de tels modèles, généralement d'aspect géométrique en un sens large et, ce faisant, elles ont eu ou bien recours à des disciplines mathématiques déjà développées, ou bien ont puissamment contribué au développement de nouvelles structures mathématiques.* —André Lichnerowicz, *in the opening lecture of the meeting "Geometry and Physics" in Florence, 12–15 October 1982* [48].

In this dissertation I shall examine a geometric approach to the study of differential equations, in particular those which are associated with the calculus of variations. The main tool which I shall use in this approach is the family of jet bundles associated with a locally trivial fibred manifold.

The geometrical study of differential equations has a long and honourable history. The use of the group of symmetries of an equation was, from this point of view, an attempt to ignore those aspects of a differential equation which were dependent upon the choice of coordinates in which the differential relationship was expressed; the principle of relativity was perhaps a more explicitly geometric approach. With the coming of formalism in mathematics it was seen that the "differentiable change of coordinates" under which these relationships were supposed to be invariant was nothing less than an assertion that the study of these equations should be undertaken on a differentiable manifold rather than in Euclidean space. The use of manifolds also allows global topological properties of the relationships between the variables to be taken into consideration.

In the past forty years there has been a considerable expansion in the use of these geometrical techniques. Two major areas of attack have been the equations of classical mechanics and those of field theories over space-time; in mechanics, the tangent and cotangent manifolds have been used for defining Lagrangian and Hamiltonian systems respectively, whereas in field theories a major area of interest has been in the use of connections on principal fibre bundles. Another application has been to the study of completely integrable evolution equations and their soliton solutions. A common theme has been the involvement of the calculus of variations, and one of the major problems of the past decade has been to extend the constructions used for the ordinary differential equations associated with variational principles in mechanics so that they may be applied to partial differential equations. There have also been studies of higher-order systems where the Lagrangian function (whose integral is to be made stationary) involves derivatives higher than the first.

The most important geometrical tool for this investigation is the jet bundle. One of my main objectives in this dissertation will be to emphasize the parallels between the geometry of tangent bundles and the geometry of jet bundles; one would expect these parallels to be close, because a tangent bundle is really just a particular kind of jet bundle.

There are essentially two types of intrinsic geometrical structure on a jet bundle, corresponding to the fibration of a jet manifold over the two manifolds of independent and dependent variables. The first of these is now quite well understood; it may be defined by an operation of "horizontalisation" on differential forms and leads to the contact structure and to the natural connection on the infinite jet manifold. The corresponding horizontal and vertical differentials play an important part in the study of the calculus of variations and of the inverse problem (namely the problem of when a system of differential equations is derived from a variational principle). There is a vestigial form of this structure on tangent bundles, namely the total time derivative operator on differential forms.

The second type of geometrical structure is one which is much more familiar on tangent bundles, namely the almost tangent structure (or "vertical endomorphism") and its generalisations to higher-order tangent bundles. This type of structure has not so far been extended to more general jet bundles, except in a local coordinate-dependent way. One of the main original aspects of this dissertation is to show how this extension may be carried out. As an application of this structure I shall demonstrate how to construct a Cartan form in higher-order field theories; this is a generalisation of the geometrical construction originally used by Cartan in classical mechanics.

To continue the parallel between jet bundles and tangent bundles, I shall examine the properties of "jet fields"; these are the jet bundle version of vector fields, and when combined with the horizontalisation operator yield vector-valued 1-forms which define Cartan-Ehresmann connections. One application of this construction is to second-order jet fields (defined just like second-order differential equation fields in tangent bundle geometry); I shall show how the equation which relates the (time-dependent) Euler-Lagrange field to the Cartan 2-form may be generalised to a relationship between the vector-valued 1-form corresponding to an Euler-Lagrange jet field and the corresponding Cartan $(m+1)$-form. I shall also explain how to define jet fields along bundle maps (these correspond to vector fields along maps) and show how they provide a natural setting for earlier studies of Bäcklund transformations.

In the final part of this dissertation I shall turn to the study of soliton equations and give a brief survey of some of their properties. Although there are several ways of considering these equations in the context of jet bundles, I shall be particularly interested in a recent description using Kac-Moody algebras; I shall show how these algebras (and the vector fields defined on them) arise naturally when studying the total time derivative operator on the infinite tangent manifold to a semisimple Lie group, and how the generalised vertical endomorphism also makes an appearance.

The detailed structure of this dissertation is as follows. In Chapter 1, I shall review the properties of bundles and their vector fields and differential forms, and also describe the theory of derivations along bundle maps. Chapter 2 is a coherent account of the theory of jet bundles which includes a new construction (the derivations of type h_* and v_*, and the vertical bracket of vector-valued forms) and also describes infinite jets in a way which clarifies the manifold structure of J^∞ and the properties of smooth functions on this space. Chapter 3 describes the prolongations of bundle maps and of vector fields, and also introduces the notion of repeated jets. In these first three chapters I have not attempted to give a systematic survey of the literature because I have been concerned mainly to collect together the machinery which will subsequently be needed.

In Chapter 4, I define jet fields and their associated connections, and establish some of their properties. In Chapter 5 I look at the history of the almost tangent structure on a tangent manifold and explain how to generalise it to jet manifolds; I also examine the properties of this generalisation. In Chapter 6 I describe Lagrangian and Hamiltonian systems in mechanics, demonstrate a modern approach to the calculus of variations, construct a Cartan form, and explain the use of integral sections of jet fields as extremals of variational problems. Finally in Chapter 7 I consider completely integrable evolution equations, discuss their Hamiltonian structure, describe the inverse scattering transform and explain the appearance of a Kac-Moody algebra in terms of higher-order tangent manifolds.

Much of the material in these final chapters is original. Parts of Chapters 5 and 6 have been published in [63]; parts of Chapters 4 and 6 are to appear in [65]; two results in Chapter 7 are to appear in [16] in a work prepared jointly with Mike Crampin; and material from the final section of Chapter 7 has been submitted for publication.

Finally I should like to acknowledge the assistance of Mike Crampin, who has supervised this work. He has provided consistent encouragement and, besides directing me towards the study of jets in the first place, has also made numerous contributions to the geometrical study of mechanics which have motivated the extensions to field theories made within.

Chapter 1

Bundle Theory

In this introductory chapter I shall review the definitions of fibred manifolds and bundles, and consider the various manifestations of vector fields and differential forms in this context. All this material is standard; I have included it to establish the notation which will be used in the sequel. I shall also explain how the theory of derivations developed by Frölicher and Nijenhuis [29] can be extended to describe derivations along bundle maps.

I shall adopt the convention that all manifolds to be discussed are Hausdorff, second-countable, connected and of class C^∞; maps between manifolds will be C^∞. Unless otherwise stated, manifolds will be finite-dimensional and without boundary. For a general reference to elementary differential geometry I use [69].

1.1 Fibred Manifolds and Bundles

Definition 1.1.1 *A* fibred manifold *is a triple* (E, π, M) *where* E *and* M *are manifolds and* $\pi : E \longrightarrow M$ *is a surjective submersion.* E *is called the* total space, π *the* projection, *and* M *the* base manifold.

For shorthand, the fibred manifold is commonly referred to as E; however this is ambiguous as the same manifold may very well be the total space of two different fibred manifolds. My shorthand will be to use the same symbol π for the fibred manifold as for its projection.

If $p \in M$, then $\pi^{-1}(p) \subset E$ is called the *fibre over* p. Each connected component of the fibre is a closed submanifold of E. (Connectedness of E does not imply that the fibres are connected—for example, let E be a Möbius band with its waist deleted, let M be that waist, and let π be the obvious projection.)

Definition 1.1.2 *The fibred manifold* (E, π, M) *is called* locally trivial *if, for each* $p \in M$*, there is an open neighbourhood* $V_p \subset M$ *and a manifold* F_p*,* *together with a diffeomorphism* $\tau_p : \pi^{-1}(V_p) \longrightarrow V_p \times F_p$ *such that*

$$pr_1 \circ \tau_p = \pi|_{\pi^{-1}(V)}$$

(where pr_1 *is projection on the first factor).*

$$\begin{tikzcd}
\pi^{-1}(V_p) \arrow[r, "\tau_p"] \arrow[d, "\pi|_{\pi^{-1}(V_p)}"'] & V_p \times F_p \arrow[d, "pr_1"] \\
V_p \arrow[r, "id"'] & V_p
\end{tikzcd}$$

The diffeomorphism τ_p in the definition above is called a *local trivialisation* of E around p. It follows from the connectedness of M that all the manifolds F_p ($p \in M$) are diffeomorphic to a single manifold F called the *typical fibre* of π. (Note that, with my conventions about manifolds, local triviality requires connected fibres.) A locally-trivial fibred manifold will be called a *bundle.*

Some examples of bundles are:

- If M, N are manifolds, the bundle $(M \times N, pr_1, M)$, where

  $$pr_1 : M \times N \longrightarrow M$$

  is projection on the first factor. This is called a *trivial* bundle.

- Any vector bundle (where the fibres are vector spaces and the local trivialisations are required to be linear on each fibre); in particular, the tangent, cotangent and tensor bundles of a manifold.

- Any fibre bundle (where there is an action of a given Lie group G on each fibre); for example, any vector bundle with typical fibre V and group $GL(V)$, or the linear frame and coframe bundles of an m-dimensional manifold where each fibre is isomorphic to the group $GL(m, \mathbf{R})$.

- Any affine bundle (where there is an action of an associated vector bundle, on which the affine bundle is said to be modelled).

As well as bundles, there are also bundle morphisms.

Definition 1.1.3 *A* bundle morphism $(E, \pi, M) \longrightarrow (F, \rho, N)$ *is a pair of functions* $(f, \overline{f})$ *such that the following diagram is commutative:*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & F \arrow[d, "\rho"] \\ M \arrow[r, "\overline{f}"'] & N \end{tikzcd}$$

In this definition, $\overline{f}$ is called the *projection* of f, and is uniquely defined by f. One may check that it exists if, and only if, f satisfies the following condition:

$$\text{if } a, b \in E \text{ and } \pi(a) = \pi(b) \text{ then } \rho \circ f(a) = \rho \circ f(b)$$

that is, f maps fibres of π to fibres of ρ. For shorthand, I shall often refer to a map $f : E \longrightarrow F$ satisfying this condition as a bundle morphism (or sometimes a bundle map). If $f, \overline{f}$ are both embeddings then π is called a *sub-bundle* of ρ.

As one might expect, the collection of bundles and bundle morphisms forms a category. If $f : \pi \longrightarrow \rho$ and $g : \rho \longrightarrow \nu$ are both bundle morphisms then $\overline{g \circ f} = \overline{g} \circ \overline{f}$. If f is a bundle *isomorphism* then $f : E \longrightarrow F$, $\overline{f} : M \longrightarrow N$ are both diffeomorphisms (and conversely).

One of the most important concepts used in the study of bundles is that of a section. This is a generalisation of the idea of a function between manifolds and is intended to incorporate global (topological) properties.

Definition 1.1.4 *If* (E, π, M) *is a bundle then a* section *of* π *is a map* $\phi : M \longrightarrow E$ *satisfying* $\pi \circ \phi = id_M$. *A* local section *of* π *is a section of* *a sub-bundle* $(\pi^{-1}(V), \pi|_{\pi^{-1}(V)}, V)$ *where* V *is an open submanifold of* M. *The set of sections is denoted* $\Gamma(\pi)$, *the set of local sections* $\Gamma_{loc}(\pi)$.

If the bundle (E, π, M) is trivial with $E = M \times N$ then the set of functions $M \longrightarrow N$ and the set of sections $M \longrightarrow M \times N$ are in 1–1 correspondence: in fact, the function f corresponds to the section (id_M, f), which is just the *graph* of f. I shall sometimes call M the *space of independent variables*; correspondingly E will be said to contain both independent and dependent variables.

In certain circumstances it is possible to define the action of a bundle morphism on a section.

Definition 1.1.5 *If* (E, π, M) *and* (F, ρ, N) *are bundles,* $\phi : M \longrightarrow E$ *is a section of* π, *and* $f : E \longrightarrow F$ *is a bundle morphism with the additional property that* $\overline{f} : M \longrightarrow N$ *is a diffeomorphism, then the section* $\tilde{f}(\phi)$ *of* ρ *is defined to equal* $f \circ \phi \circ \overline{f}^{-1}$.

The construction of $\tilde{f}(\phi)$ may be described by a commutative diagram:

$$\begin{tikzcd}
E \arrow[r, "f"] & F \\
M \arrow[u, "\phi"] \arrow[r, "\overline{f}"'] & N \arrow[u, "\tilde{f}(\phi)"']
\end{tikzcd}$$

A similar definition is used when ϕ is a local section of π; the domain of $\tilde{f}(\phi)$ is then $\overline{f}^{-1}(\text{domain}(\phi))$. Since $\overline{g \circ f} = \overline{g} \circ \overline{f}$, it is clear that $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$.

I shall now describe several ways of generating new fibred manifolds or bundles from old ones. First, the composition of two fibred manifolds.

Definition 1.1.6 *If* (E, π, M) *and* (M, ρ, N) *are fibred manifolds then their* composition *is the fibred manifold* $(E, \rho \circ \pi, N)$.

It is straightforward to see that $\rho \circ \pi$ is a surjective submersion.

It is also possible to construct a new bundle when given two bundles with the same base manifold.

Definition 1.1.7 *The* fibred product *of the two bundles* (E_1, π_1, M) *and* (E_2, π_2, M) *is the bundle with total space* $E_1 \times_M E_2$ *defined by*

$$E_1 \times_M E_2 = \{(a_1, a_2) \in E_1 \times E_2 : \pi_1(a_1) = \pi_2(a_2)\}$$

and projection $\pi : E_1 \times_M E_2 \longrightarrow M$ *defined by* $\pi(a_1, a_2) = \pi_1(a_1) = \pi_2(a_2)$.

One may again check that π is a surjective submersion and that the bundle is locally trivial (its typical fibre is now the product of the typical fibres of π_1 and π_2).

The following construction may be used to produce, from any given bundle, a new bundle with a different base manifold.

Definition 1.1.8 *If* (E, π, M) *is a bundle and* $g : N \longrightarrow M$ *is a map, the* pull-back *of* π *over* N *is defined to be the triple* $(g^*E, g^*\pi, N)$*, where* g^*E *is the closed submanifold of* $N \times E$ *defined by*

$$g^*E = \{(p, a) \in N \times E : g(p) = \pi(a)\}$$

and the functions $g^*\pi$, $g\pi$ *are defined by*

$$\begin{aligned} g^*\pi : g^*E &\longrightarrow N; \qquad (p, a) \longmapsto p \\ g\pi : g^*E &\longrightarrow E; \qquad (p, a) \longmapsto a. \end{aligned}$$

(Note that this definition requires no special properties of the map g beyond smoothness. If (N, g, M) is itself a bundle then $g\pi = \pi^*g$, and the manifold $g^*E = \pi^*N$ is just the total space $E \times_M N$ of the fibred product of π and g.) Again the pull-back is itself a bundle, and the definition makes the following diagram commutative:

$$\begin{tikzcd} g^*E \arrow[r, "g\pi"] \arrow[d, "g^*\pi"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "g"'] & M \end{tikzcd}$$

Two special cases of this construction are of particular interest. First, if N is an (embedded) submanifold of M and $\iota : N \longrightarrow M$ is the inclusion, then $(\iota^*E, \iota^*\pi, N)$ is often called a "sub-bundle" of (E, π, M). In these circumstances $\iota\pi : (\pi(a), a) \longmapsto a$ is an injection and I shall identify ι^*E with its image in E. With this identification, $\iota^*\pi$ really is a sub-bundle of π as defined above.

Secondly, suppose that π is actually a vector bundle. Then a vector space structure may be defined on the fibres of $g^*\pi$, as follows. If $(p, a), (q, b) \in g^*E$ are in the same fibre then $g^*\pi(p, a) = g^*\pi(q, b)$ so that $p = q$, $g(p) = g(q)$ and hence $\pi(a) = \pi(b)$. The definition $(p, a) + \lambda(p, b) = (p, a + \lambda b)$ therefore makes sense; one may check that $g^*\pi$ thereby becomes a vector bundle and that the map $g\pi$ is linear on each fibre.

Given the bundle (E, π, M), the manifold E has local coordinate systems around each point. However, only some of these coordinate systems are appropriate when considering E as the total space of π. These are called *adapted* coordinate systems.

Definition 1.1.9 *If (E, π, M) is a bundle and U is a non-empty open subset of E, then a coordinate system $u : U \longrightarrow \mathbf{R}^N$ is said to be* adapted to the fibration *if there is a splitting $\mathbf{R}^N = \mathbf{R}^m \times \mathbf{R}^n$ such that:*

$$\textit{if } a, b \in U \textit{ and } \pi(a) = \pi(b) \textit{ then } pr_1 \circ u(a) = pr_1 \circ u(b)$$

where $pr_1 : \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{R}^m$.

The meaning of this definition is that points in the same fibre have their first m coordinates equal, and are distinguished by their last n coordinates.

It is always possible to find adapted coordinate systems: for if $a \in E$ and $x : V \longrightarrow \mathbf{R}^m$ is a coordinate system around $\pi(a)$, then $x \circ \pi$ may be extended to a coordinate system around a because π^* is injective. On a bundle, the extension may be constructed from a local trivialisation.

Lemma 1.1.10 *The total space of any bundle may be covered by adapted coordinate systems.*

Proof Let (E, π, M) be a bundle. Let $a \in E$, and let $p = \pi(a) \in M$. Suppose

$$\tau : \pi^{-1}(V_1) \longrightarrow V_1 \times F$$

is a local trivialisation of E around p, so that $V_1 \subset M$ and $a \in \pi^{-1}(V_1)$. Let $x : V_2 \longrightarrow \mathbf{R}^m$ be a local coordinate system around p. Define V to equal $V_1 \cap V_2$, and write τ, x for the restricted maps $\tau\,|_{\pi^{-1}(V)}$, $x\,|_V$:

$$\tau : \pi^{-1}(V) \longleftrightarrow V \times F$$

$$x : V \longrightarrow \mathbf{R}^m.$$

Now define q to equal $pr_2 \circ \tau(a) \in F$ (so that $\tau(a) = (p, q) \in V \times F$). Let $w : W \longrightarrow \mathbf{R}^n$ be a local coordinate system around q, so that $W \subset F$. Define U to equal $\tau^{-1}(V \times W)$, and define $u : U \longrightarrow \mathbf{R}^m \times \mathbf{R}^n$ by

$$u(b) = (x \circ \pi(b), w \circ pr_2 \circ \tau(b))$$

so that u is then an adapted coordinate system around a. ∎

When dealing with the component functions of an adapted coordinate system, I shall usually adopt the following notation. If x^i $(1 \leq i \leq m)$ are the coordinate functions on M then the coordinate functions on E will be labelled

$$(x^i, u^\alpha) \qquad 1 \leq i \leq m, \quad 1 \leq \alpha \leq n$$

so that the same symbol x^i will be used both for a function $\pi(U) \longrightarrow \mathbf{R}$ and for the composite function $U \longrightarrow \pi(U) \longrightarrow \mathbf{R}$. I emphasize this point because, when dealing with vector fields and differential forms, it is sometimes unclear whether one is working upstairs or downstairs—this applies particularly when dealing with jet bundles, when there are many storeys in the skyscraper.

Coordinates can also be used when talking about sections of a bundle. If $\phi \in \Gamma_{loc}(\pi)$ is a local section then

$$\begin{aligned} x^i \circ \phi(a) &= x^i \circ \pi \circ \phi(a) && \text{(really)} \\ &= x^i(a) && \text{since } \pi \circ \phi = id \end{aligned}$$

so that the first m coordinates of $\phi(a)$ are determined by the coordinates of a. Hence only the last n coordinates are of interest in describing ϕ, and I shall define the coordinate representation of ϕ by $\phi^\alpha = u^\alpha \circ \phi$. When using coordinates, I shall (as here) be rather sloppy about restricting the domains of functions to suitably small patches. This never causes any problems.

1.2 Vector Fields on Bundles

I now move on to a review of vector fields in the context of bundles. Given any manifold M, if $\gamma : \mathbf{R} \longrightarrow M$ is a curve then $[\gamma] \in T_{\gamma(0)}M$ will denote the corresponding tangent vector. A vector field X will be regarded as a section of the tangent bundle (TM, τ_M, M), and its value at a point $p \in M$ will usually be denoted X_p. Correspondingly, a local vector field on M will be a section of the pull-back bundle $(\iota^*(TM), \iota^*(\tau_M), U)$ where U is an open submanifold of M and $\iota : U \longrightarrow M$ is the inclusion. The action of a tangent vector on a (locally-defined) function will be written $X_p f$, and the corresponding Lie derivative action of the vector field X will be denoted $\mathcal{L}_X f$ (although in later sections I shall also use the notation $d_X f$).

The set of all vector fields on M will be denoted $\mathcal{X}(M)$ rather than $\Gamma(\tau_M)$, and is both a real vector space and a module over $C^\infty(M)$; the set of all local vector fields will be denoted $\mathcal{X}_{loc}(M)$, and technically has neither of these structures. One may, of course, construct the sheaf of germs of local vector fields on M; however, I shall not make use of sheaf theory in this dissertation. Indeed, I shall often give results about $\mathcal{X}(M)$ when the extension to $\mathcal{X}_{loc}(M)$ is straightforward.

If (E, π, M) is a bundle, however, then some vector fields on the manifold E have additional properties. I shall single out the vertical vector fields and more generally those which are projectable on M; I shall also describe vector fields along the bundle map π: these are not vector fields on a manifold, although they share many of the properties of such fields. Finally I shall show how projectable vector fields act on the sections of π.

Definition 1.2.1 *If $\tau_E : TE \longrightarrow E$ is the tangent bundle to E, then the set of* vertical *vectors is defined to be*

$$V\pi = \{\xi \in TE : \pi_* \xi = 0\}.$$

$V\pi$ is a submanifold of TE, and $(V\pi, \tau_E|_{V\pi}, E)$ is a vector bundle which is a sub-bundle of τ_E and is called the *vertical bundle* of π (or sometimes, ambiguously, of E). At each point $a \in E$, the subset $V_a\pi$ of $V\pi$ defined by

$$V_a\pi = \{\xi \in V\pi : \tau_E(\xi) = a\}$$

is a vector subspace of the tangent space T_aE, and in fact is the tangent space to the fibre of π at a:

$$V_a\pi = T_a\pi^{-1}(\pi(a)).$$

In later sections I shall occasionally deal with bundles whose total spaces are infinite-dimensional manifolds, and in those circumstances it is important to know whether a given subspace of T_aE is closed. In fact $V_a\pi$ is always closed in T_aE, because π is smooth so that the linear map $\pi_*|_{V_a\pi}$ is continuous.

Definition 1.2.2 *A section of $\tau_E|_{V\pi}$ is called a* vertical vector field *on E; the vector space of vertical vector fields will be denoted $\mathcal{V}(\pi)$. Local vertical vector fields are defined in a similar way.*

So a vertical vector field is just a vector field on E which is π-related to zero. In adapted local coordinates, a vertical vector field appears as

$$X = X^\alpha \frac{\partial}{\partial u^\alpha}$$

so that the coefficients of $\frac{\partial}{\partial x^i}$ are all zero.

An example:

- If π is actually a vector bundle, then multiplication by real numbers on the fibres gives a well-defined mapping $\mathbf{R} \times E \longrightarrow E$. For each $a \in E$ there is a canonical vertical tangent vector $[t \longmapsto e^t a] \in T_aE$, and the vector field $\Delta \in \mathcal{V}(\pi)$ defined by

$$\Delta_a = [t \longmapsto e^t a]$$

  is called the *dilation field* of π. In coordinates,

$$\Delta = u^\alpha \frac{\partial}{\partial u^\alpha}.$$

A useful characterisation of vertical vector fields is given by the following lemma, which describes a condition on the Lie derivative action.

Lemma 1.2.3 *$X \in \mathcal{X}(E)$ is vertical if, and only if, for each $f \in C^\infty(M)$, $\mathcal{L}_X(\pi^* f) = 0$.*

Proof This is obtained directly from the definitions. At each $a \in E$,

$$\mathcal{L}_X(\pi^* f)(a) = X_a(\pi^* f) = \pi_* X_a(f).$$

If X is vertical then $\mathcal{L}_X(\pi^* f)(a) = 0$ for each $a \in E$, giving the condition of the lemma. Conversely, if the condition holds then $\pi_* X_a(f) = 0$ for every $f \in C^\infty(M)$, so that $\pi_* X_a = 0$. ∎

Lemma 1.2.4 *The space of vertical vector fields forms an (infinite-dimensional) Lie algebra.*

Proof This uses the result from elementary differential geometry involving the Lie bracket and π-related vector fields (see [69], p.41).

If $X, Y \in \mathcal{V}(\pi)$ then both X and Y are π-related to zero, and so $[X, Y]$ is π-related to $[0, 0]$. ∎

The complementary entity to the vertical bundle is the quotient bundle $TE/V\pi \longrightarrow E$, sometimes called the transverse bundle. I shall define this using the idea of a pull-back bundle.

Definition 1.2.5 *The* transverse bundle *of a bundle π is the bundle* $(\pi^*TM, \pi^*(\tau_M), E)$.

Lemma 1.2.6 *There is an exact sequence of vector bundles over E:*

$$0 \longrightarrow V\pi \longrightarrow TE \longrightarrow \pi^*TM \longrightarrow 0.$$

Proof $V\pi$ has been defined as a sub-bundle of TE, so the second arrow is an inclusion. For the third arrow, map $TE \longrightarrow \pi^*TM$ by $\xi \longmapsto (\tau_E\xi, \pi_*\xi)$. If $\xi \in V\pi$ then $\pi_*\xi = 0$ and conversely, so the sequence is exact at TE. Finally, if $a \in E$ and $\eta \in T_{\pi(a)}M$ then there is certainly a vector $\xi \in T_aE$ satisfying $\pi_*\xi = \eta$: for example, if—in local coordinates—

$$\eta = \eta^i \left.\frac{\partial}{\partial x^i}\right|_{\pi(a)}$$

then choose

$$\xi = \eta^i \left.\frac{\partial}{\partial x^i}\right|_a .$$

Since an arbitrary element of π^*TM may be written in this form (a, η), the third arrow is seen to be surjective and the sequence is exact. ∎

This lemma shows that $TE \cong V\pi \oplus \pi^*TM$. However, the existence of a distinguished sub-bundle of τ_E which complements the vertical bundle depends upon the availability of a *connection* on π; I shall discuss connections in Section 4.1. In a similar way, one may construct an exact sequence

$$0 \longrightarrow \mathcal{V}(\pi) \longrightarrow \mathcal{X}(E) \longrightarrow \mathcal{X}(\pi) \longrightarrow 0$$

where $\mathcal{X}(\pi)$ is the space of sections of the transverse bundle, although the proof of exactness at $\mathcal{X}(\pi)$ requires an additional argument which will be given in Lemma 1.2.9. Elements of $\mathcal{X}(\pi)$ will be called *vector fields along* π; however, I shall give a formal definition of these objects in a slightly more general context.

Definition 1.2.7 *Let M, N be manifolds, and $f : N \longrightarrow M$ a map. A* vector field along f *is a section of the pull-back bundle $(f^*TM, f^*(\tau_M), N)$.*

This definition may be used more readily by identifying each section X of $f^*(\tau_M)$ with a map $\widehat{X} : N \longrightarrow TM$ satisfying $\tau_M(\widehat{X}(p)) = f(p)$ for $p \in N$:

$$\begin{tikzcd}
f^*(TM) \arrow[r, "f(\tau_M)"] & TM \arrow[d, "\tau_M"] \\
N \arrow[u, "X"] \arrow[ur, "\widehat{X}"] \arrow[r, "f"'] & M
\end{tikzcd}$$

For given X, define $\widehat{X}$ to be $f(\tau_M) \circ X$; conversely, given $\widehat{X}$, define X by $X(p) = (p, \widehat{X}_p) \in f^*(TM)$. I shall normally use this identification and regard a vector field along f as a map $N \longrightarrow TM$. The set of all such vector fields will be denoted $\mathcal{X}(f)$; it is both a vector space and a module over $C^\infty(N)$.

If (E, π, M) is a bundle then it is natural to consider vector fields along π. In local coordinates they appear as

$$X = X^i \frac{\partial}{\partial x^i}$$

where X^i are functions on the total space E, but $\frac{\partial}{\partial x^i}$ are supposed to be vector fields on M. Where confusion is possible I shall write this as

$$X_a = X^i(a) \left. \frac{\partial}{\partial x^i} \right|_{\pi(a)} \in T_{\pi(a)}M.$$

These expressions suggest that it would also be possible to define $\mathcal{X}(\pi)$ in terms of derivations, in an analogous way to the standard definition given for vector fields on manifolds. I shall show that these are equivalent.

Lemma 1.2.8 *Let $\mathcal{D}(\pi)$ be the space of linear maps $\widetilde{X} : C^\infty(M) \longrightarrow C^\infty(E)$ satisfying*

$$\widetilde{X}(fg) = \pi^*(f)\widetilde{X}(g) + \pi^*(g)\widetilde{X}(f).$$

Then $\mathcal{D}(\pi) \cong \mathcal{X}(\pi)$.

Proof This is just a variation on the standard proof which applies to vector fields and derivations on manifolds, rather than along maps. Given a function $\widetilde{X} : C^\infty(M) \longrightarrow C^\infty(E)$ and a point $a \in E$, define $X_a : C^\infty(M) \longrightarrow \mathbf{R}$ by $X_a(f) = (\widetilde{X}f)(a)$. If $\widetilde{X}$ is linear and satisfies the derivation property above then X_a is clearly linear and satisfies

$$\begin{aligned} X_a(fg) &= \widetilde{X}(fg)(a) \\ &= \left(\pi^*(f)\widetilde{X}(g) + \pi^*(g)\widetilde{X}(f)\right)(a) \\ &= f(\pi(a))X_a(g) + g(\pi(a))X_a(f) \end{aligned}$$

for $f, g \in C^\infty(M)$. Since M is a finite-dimensional C^∞ manifold, this property is sufficient to show that $X_a \in T_{\pi(a)}M$ (see [69], p.12). It can be seen in the usual way that the map $X : a \longmapsto X_a$ is C^∞ by working locally, writing X_a in terms of the basis fields $\left.\frac{\partial}{\partial x^i}\right|_{\pi(a)}$ and observing that the coefficients are $(\widetilde{X}x^i)(a)$, where x^i and therefore $\widetilde{X}x^i$ are C^∞. The reverse implication, showing that a vector field along π is a derivation, is straightforward. ∎

If $X \in \mathcal{X}(\pi)$, I shall use the notation $\mathcal{L}_X$ for this action of X on functions in $C^\infty(M)$. In fact, the Lie derivative action of a vector field along π relates it (in a non-unique way) to a vector field on the total space.

Lemma 1.2.9 *If $Y \in \mathcal{X}(\pi)$ then $\mathcal{L}_Y = \mathcal{L}_X \circ \pi^*$ for some $X \in \mathcal{X}(E)$.*

Proof I shall first demonstrate the local existence of X, and then extend the result using a partition of unity.

The local assertion is that for each $a \in E$ there is a neighbourhood U and a vector field X defined on U such that, for all $f \in C^\infty(\pi(U))$, $\mathcal{L}_Y f = \mathcal{L}_X(f \circ \pi|_U)$. The proof is similar to the proof of the last part of Lemma 1.2.6: for if

$$Y_b = Y^i(b) \left.\frac{\partial}{\partial x^i}\right|_{\pi(b)} \in T_{\pi(b)}M \qquad \text{where } b \in U$$

then one may take $X = Y^i \frac{\partial}{\partial x^i}$—that is,

$$X_b = Y^i(b) \left.\frac{\partial}{\partial x^i}\right|_b \in T_b E.$$

Then $Y_b = \pi_* X_b$, so that $\mathcal{L}_Y f(b) = \mathcal{L}_X(f \circ \pi|_U)(b)$.

To globalise this result, let U_λ be a covering of E by domains of coordinate charts, and let X_λ be the corresponding vector fields on U_λ satisfying $\mathcal{L}_Y f = \mathcal{L}_{X_\lambda}(f \circ \pi|_{U_\lambda})$. Let ϕ_λ be a partition of unity subordinate to U_λ, and put $X = \sum \phi_\lambda X_\lambda$. Then for every $b \in E$ and every $f \in C^\infty(M)$,

$$\begin{aligned}(\mathcal{L}_X \circ \pi^*)(f)(b) &= X_b(f \circ \pi) \\ &= \sum \phi_\lambda(b)(X_\lambda)_b(f \circ \pi) \\ &= \sum \phi_\lambda(b) Y_b f \\ &= Y_b f \\ &= (\mathcal{L}_Y f)(b).\end{aligned}$$

▮

Working in the opposite direction, one may also start with a vector field $X \in \mathcal{X}(E)$ and then define $\pi_* X \in \mathcal{X}(\pi)$ by $(\pi_* X)_a = \pi_* X_a \in T_{\pi(a)}M$: with this definition $\mathcal{L}_{\pi_* X} = \mathcal{L}_X \circ \pi^*$, and indeed the preceding lemma shows that the map $\pi_* : \mathcal{X}(E) \longrightarrow \mathcal{X}(\pi)$ is surjective, justifying my claim about exactness of the sequence

$$0 \longrightarrow \mathcal{V}(\pi) \longrightarrow \mathcal{X}(E) \longrightarrow \mathcal{X}(\pi) \longrightarrow 0.$$

The vector field $\pi_* X$ along π may be defined in the way I have described for any $X \in \mathcal{X}(E)$; the vector field *on the manifold* M sometimes called $\pi_* X$ exists only under the particular circumstances that X is "projectable". To examine projectable vector fields, I shall return to a consideration of the vertical bundle; in fact this is a special case of a more general construction, because each vector field on M defines an affine sub-bundle of τ_E modelled on the vector bundle $\tau_E|_{V\pi}$.

Definition 1.2.10 *If* $Y \in \mathcal{X}(M)$ *then the set of* vectors projecting to Y *is defined to be*

$$T_Y\pi = \{\xi \in TE : \pi_*(\xi) = Y_{\pi\tau_E(\xi)}\}.$$

Each fibre of $\tau_E|_{T_Y\pi}$ is a coset of the corresponding fibre of $\tau_E|_{V\pi}$, and the action which makes $\tau_E|_{T_Y\pi}$ into an affine bundle is defined by

$$\begin{aligned}V\pi \times_E T_Y\pi &\longrightarrow T_Y\pi \\ (\varsigma, \xi) &\longmapsto \varsigma + \xi.\end{aligned}$$

Definition 1.2.11 *A section of* $\tau_E|_{T_Y\pi}$ *is called a* vector field projecting to Y*; the affine space of all vector fields projecting to* Y *will be denoted* $\mathcal{X}_Y(\pi)$.

So "X projects to Y" is just another way of saying that X and Y are π-related.

The property of being a vector field projecting to Y is of course a pointwise property. However, there is normally no reason to choose a particular vector field on M (unless it is the zero field), and one usually considers the space of all vector fields on E which project to *some* vector field on M. I shall give this definition in a form which uses the fact that (TE, π_*, TM) is a bundle.

Definition 1.2.12 *A vector field X on E is called* projectable on M *if it is a bundle morphism from π to π_*:*

$$\begin{tikzcd} E \arrow[r, "X"] \arrow[d, "\pi"'] & TE \arrow[d, "\pi_*"] \\ M \arrow[r] & TM \end{tikzcd}$$

The map $\overline{X} : M \longrightarrow TM$ which makes this diagram commutative is necessarily a section of τ_M, and so a vector field on M. It is then immediate that $X \in \mathcal{X}_{\overline{X}}(\pi)$. Of course the general property of being projectable, unlike the property of projecting to a particular vector field, is *not* a pointwise property; the requirement is that the tangent vectors at all points of a given fibre must project to the same (but otherwise arbitrary) tangent vector on M, rather than to a pre-assigned tangent vector.

The following properties of projectable vector fields are essentially restatements of standard results from differential geometry.

Lemma 1.2.13 *The vector fields on E which are projectable on M form an (infinite-dimensional) Lie algebra.*

Proof The projectable vector fields certainly form a vector space. If X, $\overline{X}$ are π-related and Y, $\overline{Y}$ are π-related then so are $[X, Y]$ and $[\overline{X}, \overline{Y}]$. ∎

Lemma 1.2.14 *If X is projectable and Y is vertical then $[X, Y]$ is vertical.*

Proof $[X, Y]$ is π-related to $[\overline{X}, 0]$ by the same argument as above, and so projects to zero. ∎

Lemma 1.2.15 *If X is projectable then $\pi_* X = \overline{X} \circ \pi$.*

Proof $(\pi_* X)_a = (\pi_* \circ X)_a = (\overline{X} \circ \pi)_a$, where the first equality is just the definition of $\pi_* X$ and the second is the definition of $\overline{X}$. ∎

Lemma 1.2.16 *If X is projectable then $\mathcal{L}_X \circ \pi^* = \pi^* \circ \mathcal{L}_{\overline{X}}$.*

Proof A sequence of basic manipulations, using the definition of $\overline{X}$. If $f \in C^\infty(M)$, $a \in E$ then

$$\begin{aligned}
(\mathcal{L}_X \circ \pi^*)(f)(a) &= X_a(\pi^*(f)) \\
&= \pi_* X_a(f) \\
&= \overline{X}_{\pi(a)}(f) \\
&= \mathcal{L}_{\overline{X}} f(\pi(a)) \\
&= (\pi^* \circ \mathcal{L}_{\overline{X}})(f)(a).
\end{aligned}$$

∎

Proposition 1.2.17 *If $X \in \mathcal{X}(E)$ is a complete vector field with flow ψ then X is projectable to $\overline{X}$ if, and only if, for each $t \in \mathbf{R}$ the diffeomorphism ψ_t defines a bundle isomorphism $(\psi_t, \overline{\psi}_t)$ from π to itself, where $\overline{\psi}$ is the flow of $\overline{X}$.*

Proof Suppose first that each ψ_t gives rise to a bundle isomorphism $(\psi_t, \overline{\psi}_t)$; the proof that X is projectable then just uses the definitions. For each $a \in E$,

$$\begin{aligned}
\pi_* X_a &= \pi_*[t \longmapsto \psi_t(a)] \\
&= [t \longmapsto \pi(\psi_t(a))] \\
&= [t \longmapsto \overline{\psi}_t(\pi(a))]
\end{aligned}$$

so that the tangent vector $\psi_* X_a$ depends only on the point $\pi(a) \in M$: hence X is a bundle map from π to π_*. The projection of the vector field X to a map $\overline{X} : M \longrightarrow TM$ then satisfies $\overline{X}_{\pi(a)} = [t \longmapsto \overline{\psi}_t(\pi(a))]$ by definition; furthermore $\overline{X}_{\pi(a)} \in T_{\pi(a)}M$ because $\overline{\psi}_0 = id_M$, so that $\overline{X}$ is a vector field on M.

The proof of the converse assertion relies on the uniqueness of integral curves. Suppose that X is projectable to $\overline{X}$. Given $a \in E$, the integral curve of X through a is $t \longmapsto \psi_t(a)$ and so the integral curve of $\overline{X}$ through $\pi(a)$ is $t \longmapsto \pi(\psi_t(a))$; consequently

$$\pi(\psi_t(a)) = \overline{\psi}_t(\pi(a))$$

and so, for each t, $(\psi_t, \overline{\psi}_t)$ is a bundle map. It is a bundle isomorphism because both ψ_t and $\overline{\psi}_t$ are diffeomorphisms. ∎

A similar result holds when the vector field X is not complete; however the domain and image of the bundle isomorphism are then only sub-bundles of π.

If (x^i, u^α) is an adapted coordinate system on E, then the local coordinate presentation of a projectable vector field X is

$$X = X^i \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial u^\alpha}$$

where $X^i, X^\alpha \in C^\infty(E)$ and the functions X^i are constant along the fibres of π (so that there are functions $\overline{X}^i \in C^\infty(M)$ with $X^i = \overline{X}^i \circ \pi$). The local presentation of the projected vector field $\overline{X}$ is then

$$\overline{X} = \overline{X}^i \frac{\partial}{\partial x^i}$$

where the x^i are now coordinates on M. Most people ignore the distinction between X^i and $\overline{X}^i$ when working in coordinates.

Finally in this section I shall show how a vector field on E may act on a section ϕ of π to give a vector field along ϕ (which, since both π and τ_E are bundles, may be regarded as a section of the composite bundle $\pi \circ \tau_E$). I shall initially give a definition in terms of flows.

Definition 1.2.18 *The action of $\mathcal{X}(E)$ on $\Gamma_{loc}(\pi)$ is the map $(X, \phi) \longmapsto \tilde{X}\phi$ given pointwise by*

$$(\tilde{X}\phi)_p = [t \longmapsto \psi_t \circ \phi \circ \pi \circ \psi_{-t} \circ \phi(p)] \in T_{\phi(p)}E$$

where ψ_t is the flow of X in a neighbourhood of $\phi(p) \in E$.

If X is projectable then this slightly lengthy expression may be simplified.

Lemma 1.2.19 *If $X \in \mathcal{X}(E)$ projects to $\overline{X} \in \mathcal{X}(M)$ and $\overline{\psi}_t$ is the flow of $\overline{X}$ in a neighbourhood of $p \in M$ then*

$$(\tilde{X}\phi)_p = [t \longmapsto \tilde{\psi}_t(\phi)(p)].$$

Proof From the property of $(\psi_t, \overline{\psi}_t)$ as a bundle map and the definition of $\tilde{\psi}_t$:

$$\begin{aligned} \psi_t \circ \phi \circ \pi \circ \psi_{-t} \circ \phi &= \psi_t \circ \phi \circ \overline{\psi}_{-t} \circ \pi \circ \phi \\ &= \tilde{\psi}_t(\phi). \end{aligned}$$

∎

If X is actually vertical then there is a further simplification.

Lemma 1.2.20 *If* $X \in \mathcal{V}(\pi)$ *then* $\widetilde{X}\phi = X \circ \phi$.

Proof Directly from the definitions:

$$\widetilde{X}\phi_p = [t \longmapsto \widetilde{\psi}_t(\phi)(p)] = [t \longmapsto \psi_t \circ \phi(p)] = X_{\phi(p)}.$$

∎

Another interpretation of this construction is that it defines a vector field on the submanifold $\mathrm{Im}(\phi) \subset E$; given my definition 1.2.18 the latter vector field should technically be denoted $\widetilde{X}\phi \circ \pi|_{\mathrm{Im}(\phi)}$. It is evident from Lemma 1.2.20 that if X is vertical then $\widetilde{X}\phi \circ \pi$ is just the restriction of X to $\mathrm{Im}(\phi)$. Furthermore, $\widetilde{X}\phi$ is always vertical over M, and indeed from Definition 1.2.18 one may immediately write

$$(\widetilde{X}\phi)_p = X_{\phi(p)} - \phi_*\pi_* X_{\phi(p)}.$$

One might therefore ask whether it would be possible to obtain a vertical vector field from an arbitrary vector field X on E by mapping each tangent vector $X_a \in T_a E$ to $(\widetilde{X}\phi)_{\pi(a)}$, where ϕ is a local section satisfying $\phi(p) = a$. The trouble with this idea is that such a mapping of tangent vectors involves ϕ_* and therefore depends not just on the value of ϕ at $\pi(a)$ but also on its first derivatives at that point. In fact, this construction introduces the idea of a *connection* which I shall explain in more detail in subsequent chapters.

1.3 Differential Forms on Bundles

By analogy with vector fields, I shall regard a differential 1-form on the manifold M as a section $\sigma : M \longrightarrow T^*M$ of the cotangent bundle. Similarly, a k-form will be a section $\omega : M \longrightarrow \bigwedge^k T^*M$ of the exterior power bundle. The contraction of a form ω with a vector field X will usually be written $X \lrcorner\, \omega$ and the Lie derivative of a form by a vector field as $\mathcal{L}_X\omega$, although when these are considered in Section 1.4 as derivations the alternative notations $i_X\omega$, $d_X\omega$ will also be employed. The formula $\mathcal{L}_X\omega = X \lrcorner\, d\omega + d(X \lrcorner\, \omega)$, well-known from basic differential geometry, will also appear again in this latter context.

If (E, π, M) is a bundle then some cotangent vectors and differential forms have additional properties. In this section I shall show how the counterpart to a vertical tangent vector (or vector field) is a horizontal cotangent vector (or 1-form) and how the counterpart to a vector field along a bundle map is a "vertical" 1-form, which may also be considered (when the base manifold M is orientable) as an m-horizontal $(m+1)$-form. Finally I shall show how similar ideas may be applied to vector-valued forms.

Definition 1.3.1 *If $\tau_M^* : T^*M \longrightarrow M$ is the cotangent bundle to M then the bundle of* cotangent vectors horizontal over M *is the pull-back bundle* $(\pi^*T^*M, \pi^*(\tau_M^*), E)$.

As in Definition 1.2.7, I shall actually work with a slightly different realisation of this bundle. Each element of π^*T^*M (the total space of the pullback bundle) is of the form $(a, \eta) \in E \times_M T^*M$; I shall identify each such element with $\pi^*\eta \in \pi^*T^*M$ (the space of cotangent vectors on M pulled back to E). The coincidence of notation for these two conceptually different spaces is therefore not a problem.

Definition 1.3.2 *A section σ of the bundle $\pi^*(\tau_M^*)$ is called a* 1-form on E horizontal over M.

Another name for a horizontal 1-form is a *semi-basic* 1-form.

I shall use the notation $\bigwedge^k E$ for the space of k-forms on E and $\bigwedge E$ for the algebra of all differential forms on E. That is,

$$\bigwedge^k E = \Gamma(\bigwedge^k T^*E \longrightarrow E) :$$

there will rarely be confusion between the module of forms and the manifold of exterior powers of cotangent vectors. I shall also adopt the notation $\bigwedge_0^1 \pi$ for the 1-forms on E horizontal over M (the reason for this will become apparent in a moment). It is clear that $\bigwedge_0^1 \pi$ is a vector space and that it is the module generated over $C^\infty(E)$ by $\{\pi^*\sigma : \sigma \in \bigwedge^1 M\}$; the following lemma shows that it is the annihilator of $\mathcal{V}(\pi)$ under the operation of contraction.

Lemma 1.3.3 *If $\sigma \in \bigwedge^1 E$ then $\sigma \in \bigwedge_0^1 \pi$ if, and only if, for every vertical vector field $X \in \mathcal{V}(\pi)$,*

$$X \lrcorner \sigma = 0.$$

Proof The structure of this proof is similar to that of the proof in basic differential geometry that the module dual to $\mathcal{X}(E)$ is $\bigwedge^1 E$. If $\sigma \in \bigwedge_0^1 \pi$ then, for each $a \in E$, $\sigma_a = \pi^*\eta$ for some $\eta \in T^*_{\pi(a)}M$. Then if $X \in \mathcal{V}(\pi)$,

$$\begin{aligned} (X \lrcorner \sigma)_a &= \sigma_a(X_a) \\ &= \pi^*\eta(X_a) \\ &= \eta(\pi_* X_a) \\ &= 0. \end{aligned}$$

Conversely, suppose $\sigma \in \bigwedge^1 E$ and that $X \lrcorner \sigma = 0$ for every $X \in \mathcal{V}(\pi)$. Let $a \in E$. For each $\varsigma \in V_a\pi$ there is a vertical vector field $X \in \mathcal{V}(\pi)$ such that $X_a = \varsigma$; X may be constructed by, for example, writing ς in local coordinates, choosing smooth functions whose values at a are those coordinates, and then extending the local vertical vector field so defined to the whole of E by using a bump function. Then

$$\sigma_a(\varsigma) = \sigma_a(X_a) = (X \lrcorner \sigma)_a = 0.$$

Define a cotangent vector $\eta \in T^*_{\pi(a)}M$ by, for $\overline{\xi} \in T_{\pi(a)}M$,

$$\eta(\overline{\xi}) = \sigma_a(\xi)$$

where $\xi \in T_aE$ satisfies $\pi_*\xi = \overline{\xi}$; if $\pi_*\xi_1 = \pi_*\xi_2 = \overline{\xi}$ then $\pi_*(\xi_1 - \xi_2) = 0$ so that $\xi_1 - \xi_2 \in V_a\pi$ and therefore $\sigma_a(\xi_1) = \sigma_a(\xi_2)$. Then for any $\xi \in T_aE$,

$$\sigma_a(\xi) = \eta(\pi_*\xi) = \pi^*\eta(\xi)$$

so that $\sigma_a = \pi^*\eta$ and therefore $\sigma \in \bigwedge_0^1 \pi$. ∎

In local coordinates, an element $\sigma \in \bigwedge^1 E$ may be written

$$\sigma = \sigma_i dx^i + \sigma_\alpha du^\alpha.$$

If $\sigma \in \bigwedge_0^1 \pi$ then

$$\sigma = \sigma_i dx^i$$

so that there are no terms in du^α; however the functions σ^i are elements of $C^\infty(E)$.

A similar idea may be applied to k-forms.

Definition 1.3.4 *A section of the bundle $\bigwedge^k \pi^* T^*M \longrightarrow E$ is called a k-form on E* horizontal over M.

I shall denote the space of horizontal k-forms by $\bigwedge_0^k \pi$ and the algebra of all horizontal forms by $\bigwedge_0 \pi$.

Lemma 1.3.5 *If $\theta \in \bigwedge^k E$ then $\theta \in \bigwedge_0^k \pi$ if, and only if, for every $X \in \mathcal{V}(\pi)$, $X \lrcorner \theta = 0$.*

Proof Similar to the proof of Lemma 1.3.3. ∎

In local coordinates a horizontal k-form is written

$$\theta = \theta_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \qquad i_1 < \ldots < i_k.$$

I have used the notation $\bigwedge_0^k \pi$ because it generalises to "partly horizontal" k-forms, where $\bigwedge_r^k \pi$ denotes the space of $(k-r)$-horizontal k-forms.

Definition 1.3.6 *A section of the bundle $(\bigwedge^r T^*E \wedge \bigwedge^{(k-r)} \pi^* T^*M) \longrightarrow E$, $(1 \leq r \leq k-1)$ is a k-form on E which is called $(k-r)$*-horizontal over M.

Lemma 1.3.7 *If $\theta \in \bigwedge^k E$ then $\theta \in \bigwedge_r^k \pi$, $(1 \leq r \leq k-1)$ if, and only if, for every $X \in \mathcal{V}(\pi)$, $X \lrcorner \theta \in \bigwedge_{r-1}^{k-1} \pi$.*

Proof Again similar to the proof of Lemma 1.3.3, but this time using multilinear algebra to demonstrate that $\theta_a \in \bigwedge^r T_a^* E \wedge \bigwedge^{(k-r)} \pi^* T_{\pi(a)}^* M$. ∎

This construction defines a filtration on the space of k-forms on E:

$$\textstyle\bigwedge_0^k \pi \subset \bigwedge_1^k \pi \subset \ldots \subset \bigwedge_{k-1}^k \pi \subset \bigwedge^k E$$

where if $r < k - \dim M$ then $\bigwedge_r^k \pi = \{0\}$. In local coordinates, a form $\theta \in \bigwedge_r^k \pi$ may be written

$$\theta = \theta_{\alpha_1 \ldots \alpha_s i_{s+1} \ldots i_k} du^{\alpha_1} \wedge \ldots \wedge du^{\alpha_s} \wedge dx^{i_{s+1}} \wedge \ldots \wedge dx^{i_k}$$

$$0 \leq s \leq r$$

$$\alpha_1 < \ldots < \alpha_s$$

$$i_{s+1} < \ldots < i_k$$

and so a form in $\bigwedge_r^k \pi$ contains r (or fewer) du^α's in each term of its coordinate expression. Note that, without the additional structure of a connection, there is no distinguished complement of $\bigwedge_r^k \pi$ in $\bigwedge_{r+1}^k \pi$.

I shall now extend the idea of contraction from vector fields on manifolds to vector fields along maps.

Definition 1.3.8 *If $X \in \mathcal{X}(\pi)$ and $\sigma \in \bigwedge^1 M$, define $X \lrcorner \sigma \in C^\infty(E)$ by, for $a \in E$,*

$$X \lrcorner \sigma(a) = \sigma_{\pi(a)}(X_a).$$

Definition 1.3.9 *If $X \in \mathcal{X}(\pi)$ and $\sigma \in \bigwedge_0^1 \pi$, define $X \lrcorner \sigma \in C^\infty(E)$ by, for $a \in E$,*

$$X \lrcorner \sigma(a) = \eta(X_a)$$

where $\eta \in T_{\pi(a)}^* M$ satisfies $\pi^* \eta = \sigma_a$.

In local coordinates, if $X = X^i \frac{\partial}{\partial x^i}$ and $\sigma = \sigma_j dx^j$ where $X^i, \sigma^j \in C^\infty(E)$ then

$$X \lrcorner \sigma = X^i \sigma_i.$$

If $\theta \in \bigwedge^k M$ or $\theta \in \bigwedge_0^k \pi$ then $X \lrcorner \omega$ is defined in a similar way.

Lemma 1.3.10 *$\bigwedge_0^1 \pi$ is isomorphic to the module dual to $\mathcal{X}(\pi)$.*

Proof The pairing $(X, \sigma) \longmapsto X \lrcorner \sigma$ clearly gives an isomorphism of $\bigwedge_0^1 \pi$ with a submodule of the dual of $\mathcal{X}(\pi)$; that it is the whole of this module follows from an argument similar to that used when showing that $\bigwedge^1 M$ is the dual of $\mathcal{X}(M)$: see [69], p.64. ∎

As well as considering horizontal cotangent vectors and differential forms, some authors use vertical cotangent vectors [39,40]. However, these entities don't bear the same relation to ordinary cotangent vectors as vertical tangent vectors do to ordinary tangent vectors: I shall show how they fit into the picture.

Definition 1.3.11 *The dual bundle $V^*\pi \longrightarrow E$ of $V\pi \longrightarrow E$ is called the* vertical cotangent bundle to E.

Specifically, let $V_a\pi$ be the vertical tangent space to E at a. The *vertical cotangent space to E at a* is then the dual space $V_a^*\pi$, the total space $V^*\pi$ is $\bigcup_{a \in E} V_a^*\pi$, and the projection $V^*\pi \longrightarrow E$ is defined by $\eta \longmapsto a$ for $\eta \in V_a^*\pi$.

Since $V_a\pi$ is a subspace of T_aE, $V_a^*\pi$ is the image of T_a^*E under the transpose of the inclusion map. Without additional structure such as a connection on π, $V_a^*\pi$ does not make a canonical appearance as a subspace of T_a^*E. Correspondingly, $V^*\pi \longrightarrow E$ is a quotient bundle of $T^*E \longrightarrow E$ rather than a sub-bundle. Sections of $V^*\pi \longrightarrow E$ will be called *vertical 1-forms*, and the collection of all such sections will be denoted $\mathcal{V}^*(\pi)$; it is a real vector space and a module over $C^\infty(E)$. The projection $\bigwedge^1 \pi \longrightarrow \mathcal{V}^*(\pi)$ for differential forms then corresponds to the projection $\mathcal{X}(E) \longrightarrow \mathcal{X}(\pi)$ for vector fields. In fact the relationship between these various constructions may be summarised by the following two diagrams, each containing a pair of short exact sequences, where the dashed vertical lines indicate pairs of mutually dual vector bundles over E or mutually dual $C^\infty(E)$ modules. In addition $V\pi$ and π^*T^*M are annihilators of each other, and correspondingly so are $\mathcal{V}(\pi)$ and $\bigwedge_0^1 \pi$.

$$
\begin{tikzcd}
0 \arrow[r] & V\pi \arrow[r] \arrow[d, dash, dashed] & TE \arrow[r] \arrow[d, dash, dashed] & \pi^*TM \arrow[r] \arrow[d, dash, dashed] & 0 \\
0 & V^*\pi \arrow[l] & T^*E \arrow[l] & \pi^*T^*M \arrow[l] & 0 \arrow[l]
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}(\pi) \arrow[r] \arrow[d, dash, dotted] & \mathcal{X}(E) \arrow[r] \arrow[d, dash, dotted] & \mathcal{X}(\pi) \arrow[r] \arrow[d, dash, dotted] & 0 \\
0 & \mathcal{V}^*(\pi) \arrow[l] & \bigwedge^1 E \arrow[l] & \bigwedge_0^1 \pi \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

In coordinates, if $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u^\alpha}$ form a local basis for the vector fields on E, then $\frac{\partial}{\partial u^\alpha}$ form a basis for the vertical vector fields. Writing $\overline{du}^\alpha$ for the dual basis of vertical 1-forms, the coordinate presentation of a vertical 1-form is

$$\theta = \theta_\alpha \overline{du}^\alpha$$

In this manifestation, $\overline{du}^\alpha$ is just the coset $du^\alpha + \bigwedge_0^1 \pi$. However there is a different interpretation of vertical 1-forms which may be used when the base manifold is orientable.

Lemma 1.3.12 *If M is orientable with a given volume form then $\mathcal{V}^*(\pi)$ is canonically isomorphic to $\bigwedge_1^{m+1} \pi$.*

Proof Let Ω denote the volume form on M, and also its pull-back to E under π. Suppose $\theta \in \bigwedge_1^{m+1} \pi$. Then for each $X \in \mathcal{V}(\pi)$, $X \lrcorner\, \theta \in \bigwedge_0^m \pi$ and so $X \lrcorner\, \theta = f_X \Omega$ for some $f_X \in C^\infty(E)$. One may therefore define $\overline{\theta} \in \mathcal{V}^*\pi$ by

$$< X, \overline{\theta} > \quad = \quad f_X$$

where angle brackets denote the pairing of $\mathcal{V}(\pi)$ with $\mathcal{V}^*(\pi)$.

This correspondence is evidently linear. To see that it is bijective, note first that θ may be written as $\sigma \wedge \Omega$ where $\sigma \in \bigwedge^1 E$: for in a coordinate neighbourhood U using coordinates (x^i, u^α) satisfying $dx^1 \wedge \ldots \wedge dx^m = \Omega$ let

$$\sigma_U = (-1)^{\frac{1}{2}m(m+1)} \frac{\partial}{\partial x^1} \lrcorner \ldots \lrcorner \frac{\partial}{\partial x^m} \lrcorner\, \theta$$

and then construct σ globally on E using a partition of unity. (Of course, the 1-form σ constructed in this way is not unique.) Then

$$f_X \Omega = X \lrcorner\, \theta = (X \lrcorner\, \sigma)\Omega$$

as X is vertical, so that $X \lrcorner\, \sigma = f_X$. If $\overline{\theta_1} = \overline{\theta_2}$ then, for all $X \in \mathcal{V}(\pi)$, $X \lrcorner\, \sigma_1 = X \lrcorner\, \sigma_2$ so that $\sigma_1 - \sigma_2 \in \bigwedge_0^1 \pi$ and hence $(\sigma_1 - \sigma_2) \wedge \Omega = 0$, showing that the correspondence is injective. On the other hand, given any element $\overline{\theta} \in \mathcal{V}^*(\pi)$, there is certainly an ordinary 1-form $\sigma \in \bigwedge^1 E$ satisfying $X \lrcorner \sigma = < X, \overline{\theta} >$ for every $X \in \mathcal{V}(\pi)$; then define θ to equal $\sigma \wedge \Omega$, showing that the correspondence is also surjective. ∎

Of course, all this looks obvious in coordinates: if the vertical 1-form $\overline{\theta} = \theta_\alpha \overline{du}^\alpha$ then $\theta = \theta_\alpha du^\alpha \wedge \Omega$, and conversely.

In the final part of this section I shall consider vector-valued forms defined in the context of bundles. There are three different bundles of tangent vectors over E, namely $V\pi$, TE and π^*TM, and similarly three different bundles of cotangent vectors. One may therefore construct nine different types of vector-valued 1-form, and a correspondingly larger number of different types of vector-valued k-form. However I shall restrict attention to just two kinds of vector-valued form, depending roughly on whether the vector field part or the differential form part is projected along π.

Definition 1.3.13 *A* vector-valued k-form on E horizontal over M *is a section of the tensor product bundle* $\pi^* \bigwedge^k T^*M \otimes TE \longrightarrow E$

One may check that if R is a vector-valued k-form on E then R is horizontal over M if, and only if, for every $X \in \mathcal{V}(\pi)$, $X \lrcorner R = 0$. The module of all horizontal vector-valued k-forms may be identified with $\bigwedge_0^k \pi \otimes \mathcal{X}(E)$. In particular, a connection on the bundle π may be considered as a horizontal vector-valued 1-form R satisfying $R \lrcorner \sigma = \sigma$ whenever $\sigma \in \bigwedge_0^1 \pi$. I shall consider such entities in Section 4.1.

Definition 1.3.14 *A* vector-valued k-form along π *is a section of the tensor product bundle* $\bigwedge^k T^*E \otimes \pi^*TM \longrightarrow E$.

Note that a vector-valued form along π is *not* a vector-valued form on E, just as a vector field along π is not a vector field on E. The module of all vector-valued k-forms along π may be identified with $\bigwedge^k E \otimes \mathcal{X}(\pi)$, so that such a form may be regarded as either a $C^\infty(E)$-linear map from $\bigwedge_0^1 \pi$ to $\bigwedge^k E$, or alternatively as an alternating $C^\infty(E)$-multilinear map from $\bigotimes^k \mathcal{X}(E)$ to $\mathcal{X}(\pi)$. These forms will be used in Section 1.4.

When $k = 1$ there is a natural vector-valued 1-form along π corresponding to the inclusion map $\iota : \bigwedge_0^1 \longrightarrow \bigwedge^1 E$ (or equivalently to its transpose $\pi_* : \mathcal{X}(E) \longrightarrow \mathcal{X}(\pi)$) which I shall denote by I. In coordinates $I = dx^i \otimes \frac{\partial}{\partial x^i}$, or to be more precise

$$I_a = (\pi^*(dx^i))_a \otimes \left.\frac{\partial}{\partial x^i}\right|_{\pi(a)}$$

1.4 Derivations on Bundles

My main intention in this section is to show how vector-valued forms along the bundle map π give rise to derivations which map (ordinary) differential forms on M to differential forms on E. However, I shall start with a digression on the two conventions which are in use for defining the wedge product of differential forms.

If $\alpha \in \bigwedge^p M$ and $\beta \in \bigwedge^q M$, some authors adopt the definition

$$(X_1, \ldots, X_{p+q}) \lrcorner (\alpha \wedge \beta) =$$

$$\frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} \varepsilon_\sigma \big((X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \lrcorner \alpha\big)\big((X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}) \lrcorner \beta\big)$$

where $X_i \in \mathcal{X}(M)$, S_{p+q} is the permutation group on $(1, \ldots, p+q)$ and ε_σ is the sign of σ. Then (for example) if α and β are 1-forms,

$$(X_1, X_2) \lrcorner (\alpha \wedge \beta) = \tfrac{1}{2}((X_1 \lrcorner \alpha)(X_2 \lrcorner \beta) - (X_2 \lrcorner \alpha)(X_1 \lrcorner \beta)).$$

By contrast, I shall adopt the alternative definition with a different numerical factor,

$$(X_1, \ldots, X_{p+q}) \lrcorner (\alpha \wedge \beta) =$$

$$\frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \varepsilon_\sigma \big((X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \lrcorner \alpha\big)\big((X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}) \lrcorner \beta\big)$$

where now for 1-forms α and β,

$$(X_1, X_2) \lrcorner (\alpha \wedge \beta) = (X_1 \lrcorner \alpha)(X_2 \lrcorner \beta) - (X_2 \lrcorner \alpha)(X_1 \lrcorner \beta)$$

and actually all wedge products may be written without fractions because there are $p!$ repeats of any given first factor $(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \lrcorner \alpha$ and similarly $q!$ repeats of any given second factor. In fact, I shall define $S_{p,q}$ to be the subgroup of S_{p+q} containing those permutations σ which satisfy $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$: these permutations are called "p, q-shuffles" in [69], p.60, where this question is discussed. By restricting attention to permutations in this subgroup, each distinct factor in the above sum appears exactly once, and I can therefore write the definition of the wedge product as

$$(X_1, \ldots, X_{p+q}) \lrcorner (\alpha \wedge \beta) =$$

$$\sum_{\sigma \in S_{p,q}} \varepsilon_\sigma \big((X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \lrcorner \alpha\big)\big((X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}) \lrcorner \beta\big).$$

The precise definition of the wedge product is important for the discussion of derivations. A derivation is an operation D on differential forms which is $\mathbf{R}$-linear, maps s-forms to $(r+s)$-forms for some fixed integer r, and satisfies the following version of Leibniz' rule,

$$D(\alpha \wedge \beta) = D\alpha \wedge \beta \pm \alpha \wedge D\beta$$

where the choice of sign depends on circumstances. (Some authors call operators where the negative sign may be taken an "antiderivation", but I shall not adopt that terminology.) The integer r is called the *degree* of the derivation: for example, d, $\mathcal{L}_X$ and i_X are derivations of degree 1, 0 and -1 respectively, where $i_X\theta$ is an alternative notation for the contraction $X \lrcorner\, \theta$. It should be clear from the digression above that my choice of convention for the wedge product also has the advantage of avoiding numerical factors which would otherwise be needed in Leibniz' rule for i_X to be admitted as a derivation.

Derivations were examined by Frölicher and Nijenhuis in [29] and related to the action of vector-valued differential forms. However, their study concerned derivations acting from a given algebra of forms $\bigwedge M$ to itself, whereas I shall also be interested in derivations mapping $\bigwedge M$ to $\bigwedge E$ along a bundle map $\pi : E \longrightarrow M$. I shall therefore describe the theory in this extended context.

Definition 1.4.1 *A* derivation along π of degree r *is an* $\mathbf{R}$*-linear map* $D : \bigwedge M \longrightarrow \bigwedge E$ *satisfying the properties*

1. *if* $\theta \in \bigwedge^s M$ *then* $D\theta \in \bigwedge^{r+s} E$;
2. *if* $\theta_1 \in \bigwedge^{s_1} M$ *and* $\theta_2 \in \bigwedge^{s_2} M$ *then*
   $D(\theta_1 \wedge \theta_2) = D\theta_1 \wedge \pi^*\theta_2 + (-1)^{rs_1}\pi^*\theta_1 \wedge D\theta_2$.

I shall define two classes of derivations, those of type i_* and those of type d_*. The model for type i_* is contraction with a vector field, and for type d_* is the Lie derivative.

Definition 1.4.2 *A derivation D along π is of type i_* if, for every* $f \in C^\infty(M) \cong \bigwedge^0 M$, $Df = 0$.

Definition 1.4.3 *A derivation D along π of degree r is of type d_* if*

$$D \circ d = (-1)^r d \circ D$$

where d on the left-hand side of this equation is exterior derivative on M, and on the right-hand side is exterior derivative on E.

I shall now use vector-valued forms along π as described in Definition 1.3.14 to construct derivations of type i_*.

Proposition 1.4.4 *If R is a vector-valued r-form along π then R determines a derivation along π of type i_* and degree $(r-1)$, denoted i_R, by the following rule:*

$$(X_1, \ldots, X_{r+s-1}) \lrcorner\, i_R\theta =$$
$$\sum_{\sigma \in S_{r,s-1}} \varepsilon_\sigma\big((X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \lrcorner\, R, \pi_* X_{\sigma(r+1)}, \ldots, \pi_* X_{\sigma(r+s-1)}\big) \lrcorner\, \theta$$

for $\theta \in \bigwedge^s M$ $(s \geq 1)$, and $i_R f = 0$ for $f \in C^\infty(M)$. Furthermore, every derivation along π of type i_* and degree $(r-1)$ is determined in this way by a unique vector-valued r-form along π.

Proof The map i_R is clearly $\mathbf{R}$-linear and of degree $(r-1)$; a (not very illuminating) combinatorial argument shows that it satisfies Leibniz' rule. Since $i_R f = 0$ it is therefore a derivation of type i_*.

Conversely, suppose D is a derivation along π of type i_* and degree $(r-1)$. Define the mapping $\check{D}$ from 1-forms on M to r-forms on E by $\check{D} = D|_{\bigwedge^1 M}$. Then if $\omega \in \bigwedge^1 M$, $f \in C^\infty(M)$,

$$\check{D}(f\omega) = (Df)\pi^*\omega + (\pi^* f)\check{D}\omega = (\pi^* f)\check{D}\omega$$

since $Df = 0$; consequently $\check{D}$ is $C^\infty(M)$-linear.

I now claim that $\check{D}$ actually defines a vector-valued r-form along π. For given $a \in E$, define the linear map $D_a : T^*_{\pi(a)}M \longrightarrow \bigwedge^r T^*_a E$ by the rule

$$D_a(\omega_{\pi(a)}) = (\check{D}\omega)_a.$$

An argument similar to that mentioned in Lemma 1.3.10 shows that this does not depend on the particular 1-form ω used to define the cotangent vector $\omega_{\pi(a)}$. The map D_a may be regarded as an element of the tensor product space $\bigwedge^r T^*_a E \otimes T_{\pi(a)}M$, and so the correspondence $\overline{D} : a \longmapsto D_a$ yields a section of the bundle $\bigwedge^r T^*E \otimes \pi^*TM \longrightarrow E$.

The final part of the proof relies on the fact that any derivation of differential forms is characterised by its action on functions and 1-forms. Since $Df = i_{\overline{D}} f = 0$ and $D\omega = i_{\overline{D}}\omega$ for $\omega \in \bigwedge^1 M$ by construction, $D = i_{\overline{D}}$; if $D = i_R$ for some other vector-valued r-form R then clearly $R = \overline{D}$. ∎

Some examples:

- Suppose $E = M$ and π is the identity map on M. Then if X is a vector field, i_X is just contraction with X, so the notation is consistent. If I is the identity vector-valued 1-form then $i_I\theta = s\theta$ for $\theta \in \bigwedge^s M$.

- For general bundles (E, π, M), if X is a vector field along π then i_X is contraction with X as specified in Definition 1.3.9. If I is the vector-valued 1-form along π defined by the inclusion $\bigwedge^1_0 \pi \longrightarrow \bigwedge^1 E$ then again $i_I\theta = s\pi^*\theta$ for $\theta \in \bigwedge^s M$.

Proposition 1.4.5 *If i_R is a derivation along π of type i_* and degree $(r-1)$, then i_R determines a derivation along π of type d_* and degree r, denoted d_R, by the rule*

$$d_R = i_R \circ d + (-1)^r d \circ i_R.$$

Furthermore, every derivation along π of type d_* and degree r is determined in this way by a unique derivation of type i_*.

Proof The map d_R is certainly $\mathbf{R}$-linear and of degree r. In addition, a straightforward calculation shows that d_R satisfies Leibniz' rule, and so is a derivation along π. Clearly $d_R \circ d = (-1)^r d \circ d_R$.

Conversely, suppose D is a derivation along π of type d_* and degree r. For each $a \in E$, define $D_a : T^*_{\pi(a)}M \longrightarrow \bigwedge^r T^*_a E$ by $D_a(df_{\pi(a)}) = (Df)_a$ where $f \in C^\infty(M)$; once again this does not depend on the particular choice of f used to define the cotangent vector $df_{\pi(a)}$. Linearity of D_a follows from $\mathbf{R}$-linearity of D, so as in Proposition 1.4.4 I can obtain a vector-valued r-form along π, denoted $\overline{D}$, satisfying $Df = i_{\overline{D}}df$; since $i_{\overline{D}}f = 0$ this gives $Df = d_{\overline{D}}f$. The commutation relation with d then shows that any derivation of type d_* is completely determined by its action on functions, and hence $D = d_{\overline{D}}$. Finally, suppose i_R is some other derivation of type i_* satisfying $D = i_R \circ d + (-1)^r d \circ i_R$. Then for any $f \in C^\infty(M)$, $i_R df = i_{\overline{D}}df$ so that $R \lrcorner\, df = \overline{D} \lrcorner\, df$ and hence for any $a \in E$, $R_a(df_{\pi(a)}) = \overline{D}_a(df_{\pi(a)})$; as a and f are arbitrary, $R = \overline{D}$. ∎

Some examples:

- Suppose $E = M$ and π is the identity map on M. Then if X is a vector field, d_X is just the Lie derivative by X. If I is the identity vector-values 1-form then $d_I\theta = d\theta$, the exterior derivative of θ.

- For general bundles (E, π, M), if X is a vector field along π then d_X defines a Lie derivative action of X; for functions, this is just the action described in Lemma 1.2.8.

Proposition 1.4.6 *Every derivation along π is the sum of two derivations, one of type i_* and one of type d_*.*

Proof If D is a derivation along π then define a derivation of type d_*, denoted d_D, by $d_D f = Df$ for $f \in C^\infty(M)$. Then $D - d_D$ is a derivation of type i_*. ∎

In the remainder of this section I shall describe a bracket operation on vector-valued forms which I shall call the Frölicher-Nijenhuis bracket. This bracket operation is discussed in [12] in the context of vector-valued forms on M: however, I shall need a more general construction. For the purposes of this construction I shall suppose the existence of two bundles, (E_1, π_1, M_1) and (E_2, π_2, M_2) and a bundle morphism (ρ_1, ρ_2) from π_1 to π_2, such that (E_1, ρ_1, E_2) and (M_1, ρ_2, M_2) are themselves both bundles:

$$\begin{tikzcd} E_1 \arrow[r, "\rho_1"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\rho_2"'] & M_2 \end{tikzcd}$$

(This situation will arise if, for example, there is a sequence of manifolds

$$\ldots \longrightarrow E_{k+1} \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \ldots$$

where each map is a bundle map; in the next chapter these will be jet manifolds and their canonical projections.)

Definition 1.4.7 *If R_1, R_2 are vector-valued r-forms along π_1, π_2 respectively, then R_1, R_2 are said to be ρ-*related *if, for each $a \in E_1$ and every $\xi_1, \ldots, \xi_r \in T_a E_1$,*

$$\rho_{2*}((R_1)_a(\xi_1, \ldots, \xi_r)) = (R_2)_{\rho_1(a)}(\rho_{1*}(\xi_1), \ldots, \rho_{1*}(\xi_r)).$$

An equivalent statement of this definition would be that R_1, R_2 are ρ-related if, for every $\sigma \in \bigwedge_0^1 \pi_2$, $\rho_1^*(R_2 \lrcorner \sigma) = R_1 \lrcorner (\rho_1^* \sigma)$. Note that R_2 (if it exists) is completely determined by R_1.

Definition 1.4.8 *If R_1, R_2 are ρ-related vector-valued r-forms along π_1, π_2 respectively, and S_1, S_2 are π-related vector-valued s-forms along ρ_1, ρ_2 respectively, then the Frölicher-Nijenhuis bracket $[R_1, S_1]$ is the vector-valued $(r+s)$-form along $\pi_2 \circ \rho_1 = \rho_2 \circ \pi_1$ defined by*

$$d_{[R_1, S_1]} = d_{R_1} \circ d_{S_2} - (-1)^{rs} d_{S_1} \circ d_{R_2}.$$

It is easy to check that $d_{[R_1,S_1]}$ as specified above is indeed a derivation along $\pi_2 \circ \rho_1$ of type d_* and degree $(r+s)$, so this definition makes sense by Propositions 1.4.4 and 1.4.5.

Example:

- Suppose π_i, ρ_i are all identity maps on a single manifold M. If R, S are both vector-valued 0-forms (that is, vector fields) then $[R,S]$ is just the ordinary Lie bracket. More generally, if just R is a vector field then $[R,S]$ is the Lie derivative $\mathcal{L}_R S$.

As an aside, I shall mention that there are two other ways of constructing a derivation by combining the R_i and the S_i. First,

$$i_{R_1} \circ i_{S_2} - (-1)^{(r-1)(s-1)} i_{S_1} \circ i_{R_2}$$

is clearly a derivation of type i_* of degree $r+s-2$. If $r = s = 0$ it is identically zero; otherwise it is a generalisation of the contraction operation between vector-valued forms. Using the notation $R_1 \overline{\wedge} S_2$ for the vector-valued $(r+s-1)$-form along $\pi_1 \circ \rho_2$ defined by its action on the vector fields on E_1:

$$\begin{aligned} &(X_1, \ldots, X_{r+s-1}) \lrcorner (R_1 \overline{\wedge} S_2) \\ = &\sum_{\sigma \in S_{r,s-1}} \varepsilon_\sigma \big((X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \lrcorner R_1, \pi_{1*} X_{\sigma(r+1)}, \ldots, \pi_{1*} X_{\sigma(r+s-1)}\big) \lrcorner S_2 \end{aligned}$$

(see [29]), this derivation may be written $i_{R_1 \overline{\wedge} S_2} - (-1)^{(r-1)(s-1)} i_{S_1 \overline{\wedge} R_2}$. Secondly,

$$i_{R_1} \circ d_{S_2} - (-1)^{r(s-1)} d_{S_1} \circ i_{R_2}$$

is also a derivation, and one finds that it is equal to $d_{R_1 \overline{\wedge} S_2} + (-1)^s i_{[R_1,S_1]}$.

For the final part of this section I shall restrict attention to vector-valued forms on a single manifold M.

Lemma 1.4.9 *The space $\bigwedge M \otimes \mathcal{X}(M)$ of all vector-valued forms on M is a graded Lie algebra under the Frölicher-Nijenhuis bracket.*

Proof The bracket operation is clearly $\mathbf{R}$-linear; it satisfies

$$[S,R] = (-1)^{rs+1}[R,S]$$

by definition. A simple calculation using this definition verifies the following version of Jacobi's identity with an appropriate combination of minus signs:

$$(-1)^{rt}[R,[S,T]] + (-1)^{rs}[S,[T,R]] + (-1)^{st}[T,[R,S]] = 0. \qquad \blacksquare$$

A particularly important case arises when both R and S are vector-valued 1-forms, and then the vector-valued 2-form $[R, S]$ is called the *Nijenhuis tensor* of R and S. I shall prove a formula for $[R, S]$ and then demonstrate the geometric importance of the special case $[R, R]$.

Lemma 1.4.10 *When the Nijenhuis tensor is regarded as acting on a pair of vector fields X, Y,*

$$\begin{aligned} & (X,Y) \lrcorner [R,S] \\ = \; & [X,Y] \lrcorner R \lrcorner S + [X \lrcorner R, Y \lrcorner S] - [X \lrcorner R, Y] \lrcorner S - [X, Y \lrcorner R] \lrcorner S \\ & + [X,Y] \lrcorner S \lrcorner R + [X \lrcorner S, Y \lrcorner R] - [X \lrcorner S, Y] \lrcorner R - [X, Y \lrcorner S] \lrcorner R. \end{aligned}$$

Proof Each side of the above equation is a vector field. I shall demonstrate equality when each side is contracted with an arbitrary exact 1-form, from which the result will follow: the proof is just a long calculation. Now

$$\begin{aligned} ((X,Y) \lrcorner [R,S]) \lrcorner df &= (X,Y) \lrcorner i_{[R,S]} df \\ &= (X,Y) \lrcorner d_{[R,S]} f \\ &= (X,Y) \lrcorner d_R d_S f + (X,Y) \lrcorner d_S d_R f, \end{aligned}$$

and I shall expand the first term in detail. By definition,

$$(X,Y) \lrcorner d_R d_S f = (X,Y) \lrcorner (i_R \circ d - d \circ i_R)(i_S df)$$

and

$$\begin{aligned} (X,Y) \lrcorner i_R d i_S df &= (X \lrcorner R, Y) \lrcorner d(S \lrcorner df) + (X, Y \lrcorner R) \lrcorner d(S \lrcorner df) \\ &= d_{X \lrcorner R}(Y \lrcorner S \lrcorner df) - d_Y(X \lrcorner R \lrcorner S \lrcorner df) \\ &\quad -[X \lrcorner R, Y] \lrcorner S \lrcorner df + d_X(Y \lrcorner R \lrcorner S \lrcorner df) \\ &\quad -d_{Y \lrcorner R}(X \lrcorner S \lrcorner df) - [X, Y \lrcorner R] \lrcorner S \lrcorner df \end{aligned}$$

whereas

$$\begin{aligned} -(X,Y) \lrcorner d i_R i_S df &= -d_X(Y \lrcorner R \lrcorner S \lrcorner df) + d_Y(X \lrcorner R \lrcorner S \lrcorner df) \\ &\quad +[X,Y] \lrcorner R \lrcorner S \lrcorner df \end{aligned}$$

so that

$$\begin{aligned} (X,Y) \lrcorner d_R d_S f &= [X,Y] \lrcorner R \lrcorner S \lrcorner df \\ &\quad -[X \lrcorner R, Y] \lrcorner S \lrcorner df - [X, Y \lrcorner R] \lrcorner S \lrcorner df \\ &\quad +d_{X \lrcorner R}(Y \lrcorner S \lrcorner df) - d_{Y \lrcorner R}(X \lrcorner S \lrcorner df). \end{aligned}$$

Similarly,

$$\begin{aligned}(X,Y) \lrcorner\, d_S d_R f &= [X,Y] \lrcorner\, S \lrcorner\, R \lrcorner\, df \\ &\quad -[X \lrcorner\, S, Y] \lrcorner\, R \lrcorner\, df - [X, Y \lrcorner\, S] \lrcorner\, R \lrcorner\, df \\ &\quad +d_{X \lrcorner\, S}(Y \lrcorner\, R \lrcorner\, df) - d_{Y \lrcorner\, S}(X \lrcorner\, R \lrcorner\, df)\end{aligned}$$

and noting that (for example)

$$\begin{aligned}d_{X \lrcorner\, R}(Y \lrcorner\, S \lrcorner\, df) - d_{Y \lrcorner\, S}(X \lrcorner\, R \lrcorner\, df) &= d_{X \lrcorner\, R} d_{Y \lrcorner\, S} f - d_{Y \lrcorner\, S} d_{X \lrcorner\, R} f \\ &= d_{[X \lrcorner\, R, Y \lrcorner\, S]} f \\ &= [X \lrcorner\, R, Y \lrcorner\, S] \lrcorner\, df\end{aligned}$$

the required equality is obtained. ∎

The Nijenhuis tensor $[R,R]$, which I shall also denote N_R, turns up all over the place; the reason for its ubiquitousness lies in its relation with the eigenspaces of R. In fact, the following proposition deserves a few preliminary remarks.

At each point $p \in M$ the vector-valued 1-form R gives rise to an endomorphism of the tangent space T_pM which may have eigenvalues and eigenspaces. I shall denote the "signature" of R at p by a multi-index $I_p \in \mathbf{N}^m$ (see Definition 2.1.1 for a brief summary of multi-index notation). Here, $I_p(j)$ is the number of distinct eigenspaces of dimension j (so that $0 \leq \sum_{j=1}^m jI_p(j) \leq m$), and I shall require the map $I : M \longrightarrow \mathbf{N}^m$ given by $p \longmapsto I_p$ to be constant. The reason for this condition is that if it holds, and if (as I always assume) M is connected, then one may define $|I|$ unique eigenfunctions λ which, at each p, yield the $|I_p|$ distinct eigenvalues $\lambda(p)$; the multiplicity of each eigenfunction will be constant. One may correspondingly define $|I|$ unique distributions Δ which, at each p, yield the $|I_p|$ distinct eigenspaces Δ_p.

Proposition 1.4.11 *Suppose the vector-valued 1-form R has constant signature I where $\sum_{j=1}^m jI(j) = m$ (so that R is diagonalisable) and that $N_R = 0$. Then each eigendistribution of R is involutive.*

Proof Let $X, Y \in \mathcal{X}(M)$ belong to the distribution Δ_λ corresponding to the eigenfunction λ. A calculation using the formula from Lemma 1.4.10 shows that

$$(R^2 - 2\lambda R + \lambda^2)[X,Y] = \tfrac{1}{2}N_R(X,Y) = 0.$$

Since R is diagonalisable, so is $R - \lambda I$, and hence

$$\ker(R - \lambda I)^2 = \ker(R - \lambda I) = \Delta_\lambda.$$

Consequently $[X,Y]$ also belongs to Δ_λ. ∎

A converse to this proposition will necessarily be rather more complicated since it will have to involve, not merely the involutiveness of each eigendistribution, but the differential way in which the distributions fit together. In fact, a general result is as follows: if R has constant signature and is diagonalisable, then $N_R = 0$ is equivalent to the pair of conditions

1. if X, Y are eigenvector fields belonging to Δ_λ, Δ_μ respectively then $[X,Y]$ belongs to $\Delta_\lambda \oplus \Delta_\mu$;

2. if λ is an eigenfunction and X is an eigenvector field belonging to Δ_μ with $\mu \neq \lambda$ then $d_X(\lambda) = 0$.

For a proof of this result, see [51].

Chapter 2

The Structure of Jet Bundles

Given any bundle, there is associated to it an infinite sequence of bundles over the same base manifold. These associated bundles are called "jet bundles", and their total spaces—the jet manifolds—may be considered as containing, not merely dependent and independent variables, but also "derivative variables". In this respect, jet manifolds are natural generalisations of tangent manifolds to situations where there is more than one independent variable or where higher derivatives are involved. (There is a slight difference in that, on a tangent manifold, the "time" variable does not appear explicitly; I shall explain this distinction in due course.)

Submanifolds of jet manifolds describe relationships between the derivative variables and the others, and in certain cases define "differential equations" whose solutions (if they exist) are sections of the original bundle. The study of these submanifolds sheds light on the integrability of these equations [60] and transformations of such a submanifold may provide more general symmetries of the corresponding equation than could be obtained by transforming just the dependent and independent variables [55]. Jet manifolds also provide a convenient setting for the study of the calculus of variations, and I shall describe this theory in Chapter 6.

Jets were first defined in 1951–2 in a sequence of papers by Ehresmann [19,22,23,21,20] although they have only really been used as a tool in mathematical physics during the past decade. There are now several accounts of various aspects of jet theory — see, for example, [17,43,44,50,60,61,67]. However the theory is complicated and notations vary between authors; furthermore I am not aware of a single account which deals with all the aspects of the theory that I wish to cover. I will therefore give a self-contained description which includes a careful definition of the collection of infinite jets as a *bona-fide* differentiable manifold. There is also a new characterisation of certain derivations on jet bundles (types h_* and v_*) and a corresponding construction of a vertical bracket for vector-valued forms on jet bundles.

2.1 Jet Manifolds and Jet Bundles

As before, (E, π, M) is a bundle. In this section I shall define the (finite) jets of a local section of π and explain how to construct the jet manifolds and the corresponding jet bundles. To represent repeated derivatives with respect to the coordinate functions on M I shall use a multi-index notation, where each element of a multi-index indicates the number of times that particular coordinate derivative has been applied. Typically, capital letters $I, J, K \ldots$ will denote multi-indexes.

Definition 2.1.1 *A* multi-index *is an m-tuple I of natural numbers. The* components *of I are denoted $I(j)$, where j is an ordinary index, $1 \leq j \leq m$. The multi-index 1_j is defined by $1_j(j) = 1$, $1_j(i) = 0$ for $i \neq j$. Addition and subtraction of multi-indexes are defined componentwise (although the result of a subtraction might not be a multi-index): $(I \pm J)(i) = I(i) \pm J(i)$. The length of a multi-index is $|I| = \sum_{i=1}^{m} I(i)$ and its factorial is $I! = \prod_{i=1}^{m}(I(i))!$.*

Definition 2.1.2 $\frac{\partial^{|I|}}{\partial x^I}$ *means* $\prod_{i=1}^{m}\left(\frac{\partial}{\partial x^i}\right)^{I(i)}$. *If $|I| = 0$ then $\frac{\partial^{|I|}}{\partial x^I}$ is the identity operator.*

My summation convention will not extend to multi-indexes: any such sum will always be indicated explicitly. However, the summation convention for ordinary indices will apply to the subscript of a multi-index such as 1_j.

As an example of the use of multi-indexes I shall prove the following useful result, a higher-order version of Leibniz' rule for partial derivatives.

Proposition 2.1.3 *If $f, g \in C^\infty(M)$ then*

$$\frac{\partial^{|I|}(fg)}{\partial x^I} = \sum_{J+K=I} \frac{I!}{J!\,K!} \frac{\partial^{|J|}f}{\partial x^J} \frac{\partial^{|K|}g}{\partial x^K}.$$

Proof By induction. The result is clearly true when $|I| = 0$; so suppose it is true whenever $|I| = r$. I shall show that it is then true for $I + 1_i$, and since every multi-index of length $r + 1$ may be written in such a form for some I and some i, the inductive step will follow.

Now

$$\begin{aligned}
\frac{\partial^{|I|+1}(fg)}{\partial x^{I+1_i}} &= \sum_{J+K=I} \frac{I!}{J!\,K!} \left(\frac{\partial^{|J|+1} f}{\partial x^{J+1_i}} \frac{\partial^K g}{\partial x^K} + \frac{\partial^{|J|} f}{\partial x^J} \frac{\partial^{|K|+1} g}{\partial x^{K+1_i}} \right) \\
&= \sum_{\substack{J+K=I+1_i \\ J(i)\geq 1}} \frac{I!}{(J-1_i)!\,K!} \frac{\partial^{|J|} f}{\partial x^J} \frac{\partial^{|K|} g}{\partial x^K} \\
&\quad + \sum_{\substack{J+K=1_i \\ K(i)\geq 1}} \frac{I!}{J!\,(K-1_i)!} \frac{\partial^{|J|} f}{\partial x^J} \frac{\partial^{|K|} g}{\partial x^K}.
\end{aligned}$$

I can combine the two separate sums by adopting the convention that, if (for example) $J(i) = 0$, then the quantity $((J-1_i)!)^{-1}$ is deemed to be zero, even though $J - 1_i$ is not a *bona fide* multi-index. This convention is just the analogue of the usual convention $((-1)!)^{-1} = 0$. So then

$$\frac{\partial^{|I|+1}(fg)}{\partial x^{I+1_i}} = \sum_{J+K=I+1_i} \left(\frac{I!}{(J-1_i)!\,K!} + \frac{I!}{J!\,(K-1_i)!} \right) \frac{\partial^{|J|} f}{\partial x^J} \frac{\partial^{|K|} g}{\partial x^K}.$$

The result now follows by considering the coefficient in parentheses on the right-hand side. If $J(i) = 0$ then the first term of this coefficient is zero by my convention and, in the second term, $K(i) = I(i) + 1$ so that

$$\frac{I!}{J!\,(K-1_i)!} = \frac{(I+1_i)!}{J!\,K!}.$$

A similar result holds if $K(i) = 0$; note that $J(i) + K(i) = I(i) + 1 > 0$. Finally if $J(i), K(i)$ are both non-zero then

$$\begin{aligned}
I! \left(\frac{1}{(J-1_i)\,!K!} + \frac{1}{J!\,(K-1_i)!} \right) &= \frac{I!}{(J-1_i)!\,(K-1_i)!} \left(\frac{1}{J(i)} + \frac{1}{K(i)} \right) \\
&= \frac{I!}{(J-1_i)!\,(K-1_i)!} \frac{(J(i)+K(i))}{J(i)K(i)} \\
&= \frac{I!}{(J-1_i)!\,(K-1_i)!} \frac{(I(i)+1)}{J(i)K(i)} \\
&= \frac{(I+1_i)!}{J!\,K!}.
\end{aligned}$$

∎

I shall also need to use multi-index notation when referring to *symmetric* covariant tensors and tensor fields; these appear when I describe the detailed bundle structure of the jet system. I shall use the notation $S^rT^*M \longrightarrow M$ for the bundle of symmetric $(0, r)$ tensors over M, and a section ξ of this bundle may be written locally in coordinates as

$$\xi = \sum_{|I|=r} \xi_I dx^I$$

where each dx^I is a symmetric product of the basis 1-forms dx^i.

I shall now define the jet of a local section of π at a point p in its domain. The method is to construct an equivalence relation on $\Gamma_{loc}(\pi)$ by asserting that two local sections are *equivalent at p to order k* if, when expressed in terms of a local coordinate chart around p, they have the same Taylor expansion to order k. The k-jet of the section at p is then the corresponding equivalence class: the k-jet thus encapsulates those properties of the section which depend only on its value and derivatives of order up to k at p. Of course, the definition I have just described may seem at first sight to depend upon a particular choice of coordinate system; the fact that it does not is a consequence of the chain rule for higher derivatives. Nevertheless, it is possible to give an equivalent definition which does not have this disadvantage (the application of the chain rule now occurs in the proof—which I shall omit—that the definitions are indeed equivalent).

Definition 2.1.4 *Suppose $\phi \in \Gamma_{loc}(\pi)$, $p \in$ domain(ϕ) and $k \in \mathbf{N}$. An element $\psi \in \Gamma_{loc}(\pi)$ is defined to be* equivalent to ϕ to order k at p *if $p \in$ domain(ψ) and, for every $f \in C^\infty(E)$ and every smooth curve $\gamma : \mathbf{R} \longrightarrow M$ satisfying $\gamma(0) = p$,*

$$\left.\frac{d^r}{dt^r}\right|_{t=0} (t \longmapsto f \circ \phi \circ \gamma(t)) = \left.\frac{d^r}{dt^r}\right|_{t=0} (t \longmapsto f \circ \psi \circ \gamma(t))$$

whenever $0 \leq r \leq k$.

Definition 2.1.5 *The* k-jet of ϕ at p *is the set of $\psi \in \Gamma_{loc}(\pi)$ which are equivalent to ϕ to order k at p, and is denoted $j^k_p\phi$.*

By taking f to be successive coordinate functions u^α around $\phi(p)$ (extended smoothly to the whole of E) and γ to be the image of successive coordinate axes in $\mathbf{R}^m$ under x^{-1} one sees immediately that if $\psi \in j^k_p\phi$ then

$$\left.\frac{\partial^{|I|}\psi^\alpha}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}\phi^\alpha}{\partial x^I}\right|_p \qquad \forall I \in \mathbf{N}^m, 0 \leq |I| \leq k.$$

Definition 2.1.6 *The set of all k-jets of local sections of π, at all points of M, is called the* k-jet manifold of π, *and denoted $J^k\pi$:*

$$J^k\pi = \{j^k_p\phi : \phi \in \Gamma_{loc}(\pi), p \in \mathrm{domain}(\phi)\}.$$

Proposition 2.1.7 *$J^k\pi$ actually is a manifold.*

Proof Let $j^k_p\phi$ be any point in $J^k\pi$, and let (x^i, u^α) be an adapted coordinate system in a neighbourhood U of $\phi(p) \in E$. For each $q \in \phi^{-1}(U)$, consider the local sections ψ of π defined at q (hence in a neighbourhood of q) and satisfying $\psi(q) \in U$, denoting this set as $\Gamma_U(\pi)$. Then let

$$U^k = \{j^k_q\psi : \psi \in \Gamma_U(\pi), q \in \mathrm{domain}(\psi)\}$$

Now define a map $u^k : U^k \longrightarrow \mathbf{R}^N$ (where $N = m + n({}^{m+k}C_k)$) by

$$u^k(j^k_q\psi) = \left(x^i(q), \psi^\alpha(q), \left.\frac{\partial^{|I|}\psi^\alpha}{\partial x^I}\right|_q \right), \qquad 1 \leq |I| \leq k.$$

This map is well-defined for equivalent $\psi \in \Gamma_U(\pi)$ by definition of $j^k_q\psi$; it is also injective for the same reason. Note that the image of U^k under u^k is actually of the form

$$W \times \mathbf{R}^M$$

where $M = n({}^{m+k}C_k - 1)$ and W is the image of U under the coordinate system (x^i, u^α): for given any element of this set, choose a polynomial function with appropriate value and derivatives mapping $\mathbf{R}^m$ to $\mathbf{R}^n$, then transfer it to a local section of π using the latter coordinate system. For any particular index α and multi-index I with $0 \leq |I| \leq k$, denote the real-valued function

$$j^k_q\psi \longmapsto \left.\frac{\partial^{|I|}\psi^\alpha}{\partial x^I}\right|_q$$

by u^α_I (so that if $|I| = 0$ then $u^\alpha_I = u^\alpha$).

I shall use the sets U^k and the maps u^k to construct an atlas on $J^k\pi$. To do this, it is sufficient to show that the transition functions on overlapping charts are C^∞. So suppose (y^j, v^β) is another adapted coordinate system in a neighbourhood of $\phi(p)$; the corresponding chart on $J^k\pi$ will be given by

$$v^k(j^k_q\psi) = \left(y^j(q), \psi^\beta(q), \left.\frac{\partial^{|J|}\psi^\beta}{\partial y^J}\right|_q \right) \qquad 1 \leq |J| \leq k$$

and the real-valued function $j^k_q\psi \longmapsto \left.\frac{\partial^{|J|}\psi^\beta}{\partial y^J}\right|_q$ will be denoted by v^β_J.

Now the v^β and y^j coordinates only depend on the u^α and x^i coordinates (and not the u_I^α), and this dependence is C^∞ by hypothesis. Since (for example)

$$\left.\frac{\partial\psi^\beta}{\partial y^j}\right|_q = \left.\frac{\partial x^i}{\partial y^j}\right|_q \left(\left.\frac{\partial v^\beta}{\partial u^\alpha}\right|_{\psi(q)} \left.\frac{\partial\psi^\alpha}{\partial x^i}\right|_q + \left.\frac{\partial v^\beta}{\partial x^i}\right|_{\psi(q)} \right)$$

the $v_{1_j}^\beta$ coordinates depend in a C^∞ way on the u^α and x^i coordinates,and in an affine way on the $u_{1_i}^\alpha$ coordinates. (This affine dependence will be referred to in Proposition 2.1.13.) Successive applications of the chain rule show that, in general, each v_J^β depends in a C^∞ way on u^α, x^i and in a polynomial way on u_I^α with $1 \leq |I| \leq |J|$; it does not depend on u_I^α with $|I| > |J|$. Consequently $v^k \circ (u^k)^{-1}$ is a C^∞ transition function on the neighbourhood $u^k(U^k \cap V^k)$ of $u^k(j_p^k\phi)$. One may check that the topology induced on $J^k\pi$ is Hausdorff, second-countable and connected; consequently $J^k\pi$ is a manifold. ∎

Corollary 2.1.8 *For each $p \in M$ the subset $J_p^k\pi$ containing all the k-jets of local sections at the particular point p is an embedded submanifold of $J^k\pi$.*

Proof If the coordinate chart (U^k, u^k) used in the proof of the proposition is restricted to $U^k \cap J_p^k\pi$ then this corresponds to keeping the x^i coordinates constant; the restricted maps therefore yield a manifold structure on $J_p^k\pi$ of dimension $n({}^{m+k}C_k)$ which is clearly embedded in $J^k\pi$. ∎

Some examples:

- The manifold $J^0\pi$ is canonically identified with E by $j_p^0\phi \longmapsto \phi(p)$.
- If π is a trivial bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$ then there are canonical identifications of $J_0^1\pi$ with TF and $J^1\pi$ with $\mathbf{R} \times TF$. TF may be considered as a set of equivalence classes of smooth curves in F, where a curve in F is a local section of $(\mathbf{R} \times F, pr_1, \mathbf{R})$ defined in a neighbourhood of the origin or indeed of any fixed point $p \in \mathbf{R}$, and two curves are equivalent if their values and first derivatives are equal at p; by taking equivalence classes for all points $p \in \mathbf{R}$ one obtains a family of copies of TF indexed by p. In coordinates, starting with the adapted coordinate system (t, q^α) on $\mathbf{R} \times F$, the coordinate system on $J^1\pi$ is $(t, q^\alpha, q_1^\alpha)$ and that on $\mathbf{R} \times TF$ is $(t; q^\alpha, \dot{q}^\alpha)$. This example is important in the study of Lagrangian systems in classical mechanics, and suggests generalisations which are investigated in subsequent chapters.
- For the same bundle π one may also consider $J_0^k\pi$ where k may be greater than one. By analogy with the case of TF I shall normally

call this the *k-th order tangent manifold* and denote it instead by T^kF; consequently $J^k\pi \cong \mathbf{R} \times T^kF$. When working in coordinates I shall normally use $(q^\alpha_{(r)})$ on T^kF to correspond to the coordinate system (t, q^α) on $\mathbf{R} \times F$, where the subscript r indicates the number of dots above the q^α (see, for example, [15,17]).

- If π is now the trivial bundle $(M \times \mathbf{R}, pr_1, M)$ then there is a canonical identification of the first jet manifold $J^1\pi$ with $T^*M \times \mathbf{R}$: for if $\phi \in \Gamma(\pi)$ write $\overline{\phi} = pr_2 \circ \phi \in C^\infty(M)$; then

$$j^1_p\phi = \{\psi : p \in \operatorname{domain}(\psi); \overline{\psi}(p) = \overline{\phi}(p); d\overline{\psi}_p = d\overline{\phi}_p\}.$$

  Consequently the mapping $J^1\pi \longrightarrow T^*M \times \mathbf{R}$ given by $j^1_p\phi \longmapsto (d\overline{\phi}_p, \overline{\phi}(p))$ is well-defined, and is plainly a diffeomorphism. There is a similar identification of $J^k\pi$ with $T^{*k}M \times \mathbf{R}$, where $T^{*k}M$ is the space of higher-order cotangent vectors described in [69]. (Note that $T^{*k}M$ and T^kM as defined in the previous example are *not* dual objects for $k > 1$; however $T^{*k}M$ *does* have a dual, which I shall need to use in Section 5.4.)

- If π is the trivial bundle $(M \times \mathbf{R}^m, pr_1, M)$ and ρ is the trivial bundle $(\mathbf{R}^m \times M, pr_1, \mathbf{R}^m)$ then one may consider the subset of $J^1\rho$ containing those 1-jets $j^1_a\psi$ where $\psi_* : T_a\mathbf{R}^m \longrightarrow T_{\psi(a)}(\mathbf{R}^m \times M)$ is non-singular; this condition is independent of a choice of representative of $j^1_a\psi$ because it depends only on the first derivatives of ψ at a. Any such ψ is "locally invertible" in that for some neighbourhood U of $pr_2 \circ \psi(a) \in M$ there is a map $\psi^{-1} : U \longrightarrow \mathbf{R}^m$ satisfying $\psi^{-1} \circ pr_2 \circ \psi(b) = b$ for all b in a suitably small neighbourhood of a. This subset of $J^1\pi$ may be canonically identified with $\mathbf{R}^m \times FM$, where $FM \longrightarrow M$ is the bundle of linear frames on M. Using this identification, the subset $\rho^{-1}_{k,1}(\mathbf{R}^m \times FM) \subset J^k\rho$ may similarly be identified with $\mathbf{R}^m \times F^kM$ where $F^kM \longrightarrow M$ is the bundle of k-th order frames.

  On the other hand, one may equally consider the subset of $J^1\pi$ containing 1-jets $j^1_p\phi$ where $\phi_* : T_pM \longrightarrow T_{\phi(p)}(M \times \mathbf{R}^m)$ is non-singular. This subset of $J^1\pi$ is canonically identified with $F^*M \times \mathbf{R}^m$, where $F^*M \longrightarrow M$ is the bundle of linear coframes on M. If $j^1_a\psi \in \mathbf{R}^m \times FM$ then $(id_M, \psi^{-1}) : U \longrightarrow M \times \mathbf{R}^m$, where ψ^{-1} is as defined above, is a "locally invertible" section of π and then $j^1_a\psi \longmapsto j^1_{pr_2 \circ \psi(a)}(\psi^{-1})$ defines a canonical identification of $\mathbf{R}^m \times FM$ and $F^*M \times \mathbf{R}^m$ which corresponds to the mapping from a frame to its dual coframe. One may define k-th order coframes in the same way as k-th order frames.

As an aside, I observe that it would have been possible to define the manifold of k-jets of *global* sections of π. However, some bundles don't have global sections (for example, any non-trivial principal fibre bundle) which would have rendered the approach pointless. On the other hand, if the bundle admits any global section at all, then it will admit a global section with the same germ at a given point as any given local section whose domain contains that point: the proof (which entails considering deformations of the global section in a neighbourhood of the point) will be omitted.

Although in general I shall not be concerned with infinite-dimensional manifolds, I should point out that a similar construction can be carried out when the typical fibre of the bundle is infinite-dimensional. The one instance in later sections when such a case will arise is when I consider the first jet manifold of π_∞, the bundle of infinite jets.

Having defined jet manifolds, I now turn to the various maps linking them together.

Definition 2.1.9 *For $k \in \mathrm{N}^+$, the map $\pi_{k,k-1} : J^k\pi \longrightarrow J^{k-1}\pi$ is defined by $\pi_{k,k-1}(j^k_p\phi) = j^{k-1}_p\phi$.*

From the definition of $j^k_p\phi$ and $j^{k-1}_p\phi$ as equivalence classes, $\pi_{k,k-1}$ is obviously well-defined.

Lemma 2.1.10 *$\pi_{k,k-1}$ is a C^∞ surjective submersion.*

Proof $\pi_{k,k-1}$ is obviously surjective. In an adapted coordinate system around $j^k_p\phi$, $\pi_{k,k-1}$ maps

$$\left(x^i(q), \psi^\alpha(q), \left. \frac{\partial^{|I|}\psi^\alpha}{\partial x^I} \right|_q \right) \qquad 1 \le |I| \le k$$

to

$$\left(x^i(q), \psi^\alpha(q), \left. \frac{\partial^{|I|}\psi^\alpha}{\partial x^I} \right|_q \right) \qquad 1 \le |I| \le k-1$$

where the codomain coordinate system is the corresponding adapted one. In these coordinates, $\pi_{k,k-1}$ is locally just a projection which ignores the coordinates corresponding to k-th order derivatives. In these coordinates the map is linear, therefore C^∞; as a linear map its derivative at each point is the map itself, so the derivative is surjective at each point and is consequently a submersion. ∎

Definition 2.1.11 *For $k-1 > l \ge 0$, define $\pi_{k,l} : J^k\pi \longrightarrow J^l\pi$ recursively by $\pi_{k,l} = \pi_{k,k-1} \circ \pi_{k-1,l}$. Define $\pi_k : J^k\pi \longrightarrow M$ by $\pi_k = \pi_{k,0} \circ \pi$.*

These maps are linked in the following way:

$$\begin{array}{ccccccccc}
\cdots & J^k\pi & \xrightarrow{\pi_{k,k-1}} & J^{k-1}\pi & \cdots & J^1\pi & \xrightarrow{\pi_{1,0}} & E \\
 & \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_{k-1}} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\
\cdots & M & \xrightarrow[id_M]{} & M & \cdots & M & \xrightarrow[id_M]{} & M
\end{array}$$

Each map is a C^∞ surjective submersion, so any given $J^k\pi$ is a fibred manifold over anything below or to the right of it (namely $J^l\pi$ with $l < k$, E, or M).

Lemma 2.1.12 *For $k > l \geq 0$, $(J^k\pi, \pi_{k,l}, J^l\pi)$ is a bundle.*

Proof As a result of the observations above, I just have to show that $\pi_{k,l}$ is locally trivial. So suppose $j^l_p\phi \in J^l\pi$. Let V be a neighbourhood of $\phi(p)$ with adapted coordinate system (x^i, u^α), and let V^l, V^k be the corresponding neighbourhoods of $j^l_p\phi$, $j^k_p\phi$ with corresponding coordinate systems. Then $V^k = \pi^{-1}_{k,l}(V^l)$, and the map

$$\tau^{k,l}_p : V^k \longrightarrow V^l \times \mathbf{R}^N$$

where $N = n\sum_{r=l+1}^{k} {}^{m+r-1}C_r$, defined by

$$\tau^{k,l}_p(j^k_q\psi) = (j^l_q\psi; u^\alpha_I(j^k_q\psi)) \qquad l+1 \leq |I| \leq k$$

satisfies the conditions for a local trivialisation. ∎

Note that, although the typical fibre of $\pi_{k,l}$ is a vector space, the bundle is not in general a vector bundle since there is no consistent way to put a vector space structure on the actual fibres to make each $\tau^{k,l}_p$ into a linear map. Another way of saying this is that the bundle transition functions $\tau^{k,l}_p \circ (\tau^{k,l}_q)^{-1}$ do not form a subgroup of the general linear group when all possible adapted coordinate systems on E are considered. However, the particular case $\pi_{k,k-1}$ does have some additional structure.

Proposition 2.1.13 *For $k > 0$, $(J^k\pi, \pi_{k,k-1}, J^{k-1}\pi)$ is an affine bundle modelled on the vector bundle $\pi^*_{k-1}(S^kT^*M) \otimes \pi^*_{k-1,0}V\pi \longrightarrow J^{k-1}\pi$.*

Proof I shall describe the affine action in terms of coordinates.

Suppose $j^k_p\phi \in J^k\pi$, and let (x^i, u^α) be an adapted coordinate system around $\phi(p) \in E$. Then (x^i, u^α_I) is an adapted coordinate system in the neighbourhood of the entire fibre $\pi_{k,k-1}^{-1}(j_p^{k-1}\phi)$, and the k-th order jet coordinate functions u_I^α where $|I| = k$ may be used as fibre coordinates.

A typical element of $\pi_{k-1}^*(S^kT^*M) \otimes \pi_{k-1,0}^*V\pi$ in the fibre above $j_p^{k-1}\phi$ may be written officially in coordinates as

$$\xi = \sum_{|I|=k} \xi_I^\alpha (j_p^{k-1}\phi, dx_p^I) \otimes \left(j_p^{k-1}\phi, \left.\frac{\partial}{\partial u^\alpha}\right|_{\phi(p)} \right)$$

or, more succinctly, as

$$\xi = \sum_{|I|=k} \xi_I^\alpha dx^I \otimes \left.\frac{\partial}{\partial u^\alpha}\right|_{j_p^{k-1}\phi}.$$

If the image of $j_p^k\phi$ under the action of ξ is denoted by $j_p^k\psi$ (it will, of course, project to the same point $p \in M$) then this action may be described in coordinates as

$$u_I^\alpha(j_p^k\psi) = u_I^\alpha(j_p^k\phi) + \xi_I^\alpha \qquad |I| = k.$$

It is clear that this action is free and transitive on the fibres of $\pi_{k,k-1}$.

I must now show that the action is independent of the choice of coordinate system. If (y^j, v^β) is another coordinate system around $\phi(p)$ and

$$\xi = \sum_{|J|=k} \overline{\xi}_J^\beta dy^J \otimes \left.\frac{\partial}{\partial v^\beta}\right|_{j_p^{k-1}\phi}$$

then

$$\overline{\xi}_J^\beta = \sum_{|I|=k} \frac{|I|!\, J!}{I!\, |J|!} \left.\frac{\partial v^\beta}{\partial u^\alpha}\right|_{\phi(p)} \left.\frac{\partial x^I}{\partial y^J}\right|_p \xi_I^\alpha$$

since ξ is a tensor. (The funny coefficient appears because I am using multi-index notation for tensors.) On the other hand,

$$v_J^\beta(j_p^k\phi) = \sum_{|I|=k} \frac{|I|!\, J!}{I!\, |J|!} \left.\frac{\partial v^\beta}{\partial u^\alpha}\right|_{\phi(p)} \left.\frac{\partial x^I}{\partial y^J}\right|_p v_I^\alpha(j_p^k\phi) + f(j_p^{k-1}\phi)$$

where the second term on the right-hand side does not depend on the k-th order derivatives of ϕ: this may be seen by successive differentiation of the transformation rule for first-order jet coordinates given in Proposition 2.1.7. Since $j_p^{k-1}\phi = j_p^{k-1}\psi$ by definition, the result follows. ∎

Corollary 2.1.14 *If π is a trivial bundle $M \times F \longrightarrow M$ then $\pi_{1,0}$ is a vector bundle.*

Proof I shall define a zero section of $\pi_{1,0}$ so that the vector bundle structure of $\pi^*T^*M \otimes V\pi \longrightarrow E$ may be transported to the affine bundle. For each $a \in M \times F$ let the constant section ϕ_a through a be defined by $\phi_a(p) = (p, pr_2(a))$, and then let $z : E \longrightarrow J^1\pi$ be defined by $z(a) = j^1_{\pi(a)}\phi_a$. ∎

An example where this corollary applies:

- If π is the trivial bundle $(\mathbf{R}\times M, pr_1, \mathbf{R})$ then $J^1\pi \cong \mathbf{R}\times TM \longrightarrow \mathbf{R}\times M$ is a vector bundle whose fibre at $(t, a) \in \mathbf{R} \times M$ is $\mathbf{R} \times T_aM$.

I can also describe the bundle structure of π_k.

Lemma 2.1.15 *For $k > 0$, $(J^k\pi, \pi_k, M)$ is a bundle.*

Proof Once again, I just have to show that π_k is locally trivial. So suppose $p \in M$, and let U be a neighbourhood of p such that

1. there is a trivialisation $\tau_p : \pi^{-1}(U) \longrightarrow U \times F$
2. there is an adapted coordinate system on $\pi^{-1}(U)$

(such a U was constructed in the proof of Lemma 1.1.10). Then the map

$$\tau^k_p : \pi_k^{-1}(U) \longrightarrow U \times F \longrightarrow \mathbf{R}^N$$

(where $N = n\sum_{r=1}^{k} {}^{m+r-1}C_r$) defined by

$$\tau^k_p(j^k_q\psi) = (\tau_p(q), u^\alpha_I(j^k_q\psi)) \quad 1 \leq |I| \leq k$$

satisfies the conditions for a local trivialisation. ∎

Lemma 2.1.16 *If π is a vector bundle then so is π_k.*

Proof The vector space structure on π_k is induced directly from that of the corresponding fibre of π. Define $j^k_p\phi + \lambda j^k_p\psi$ to equal $j^k_p(\phi + \lambda\psi)$, where $\phi + \lambda\psi$ is defined on domain$(\phi) \cap$ domain(ψ). If (x^i, u^α) is a vector bundle coordinate system on E then (x^i, u^α_I) is a vector bundle coordinate system on $J^k\pi$. ∎

As a bundle, $(J^k\pi, \pi_k, M)$ will have local sections. However, some local sections may be characterised as arising from sections of π.

Definition 2.1.17 *Let ϕ be a local section of π with domain $U \subset M$. The* k-jet extension *of ϕ is the map $j^k\phi : U \longrightarrow J^k\pi$ defined by*

$$j^k\phi(p) = j^k_p\phi.$$

Note that $\pi_k \circ j^k\phi = id_M$, so that $j^k\phi$ really is a section; also, $\pi_{k,l} \circ j^k\phi = j^l\phi$ where $k > l$. In local coordinates, $j^k\phi$ appears as

$$\left(\phi^\alpha, \frac{\partial^{|I|}\phi^\alpha}{\partial x^I}\right) \qquad 1 \leq |I| \leq k.$$

Using the identification of $J^0\pi$ with E, I shall also identify $j^0\phi$ with ϕ.

Some examples:

- Let π be the trivial bundle $(\mathbf{R} \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates (x, u) and let (x, u, u_x) be the corresponding coordinates on $J^1\pi$. (When I use coordinates without superscripts on the base manifold I shall often use a notation like u_x or u_{xxt} for the derivative coordinates which would otherwise have to be written u_{1_1} or $u_{2_1+1_2}$.) Suppose ψ and χ are the sections of π_1 defined—with the standard abuse of notation—by

$$\begin{aligned}\psi(x) &= (x, \sin x, \cos x) \\ \chi(x) &= (x, \sin x, x^2).\end{aligned}$$

  Then $\psi = j^1\phi$, where ϕ is the section of π defined by $\phi(x) = (x, \sin x)$. However, χ is not the 1-jet extension of a section of π, because its u_x coordinate is not the derivative of its u coordinate.

- More generally, if π is the trivial bundle $(M \times \mathbf{R}, pr_1, M)$ then $J^1\pi \cong T^*M \times \mathbf{R}$ so that a section ψ of π_1 may be written as a pair $(\omega, \overline{\phi})$ where $\omega \in \bigwedge^1 M$ and $\overline{\phi} \in C^\infty(M)$. Often there is no relationship between ω and $\overline{\phi}$; if, however, $\psi = j^1\phi$ for some section ϕ of π then $\overline{\phi} = pr_2 \circ \phi$ and $\omega = d\overline{\phi}$ so that ω is exact. The preceding example may be considered as a special case of this one, where—if $\psi(x) = (x, \sin x, \cos x)$—then $\overline{\phi}(x) = \sin x$ and $\omega = \cos x\, dx$.

These examples show that not all sections of π_k are k-jet extensions of sections of π, so the following lemma is worth recording.

Lemma 2.1.18 *If $\psi \in \Gamma_{loc}(\pi_k)$ then ψ is the k-jet extension of some $\phi \in \Gamma_{loc}(\pi)$ if, and only if, $j^k(\pi_{k,0} \circ \psi) = \psi$.*

Proof If $j^k(\pi_{k,0} \circ \psi) = \psi$ then let $\phi = \pi_{k,0} \circ \psi$. Conversely, if $\psi = j^k\phi$ then $j^k(\pi_{k,0} \circ \psi) = j^k(\pi_{k,0} \circ j^k\phi) = j^k\phi = \psi$. ∎

In the final part of this section I shall show how k-jet extensions may be related to differential equations.

Definition 2.1.19 *A* differential equation *on the bundle π is a fibred submanifold $(R, \pi_l|_R, M)$ of π_l with a strictly positive* order *k, where k is the largest natural number satisfying*

$$\pi_{k,k-1}^{-1}(\pi_{l,k-1}R) \neq \pi_{l,k}R.$$

The above description of the order of a differential equation is intended to concentrate attention on the case where $l = k$; the point of the definition is that the additional derivative variables u_I^α where $|I| > k$ do not provide any further information about R. A differential equation of order k may therefore be regarded as being defined on the bundle π_k.

Definition 2.1.20 *A* solution *of the differential equation $(R, \pi_k|_R, M)$ is a local section $\phi \in \Gamma_{loc}(\pi)$ satisfying $j_p^k\phi \in R$ for every $p \in \mathrm{domain}(\phi)$.*

Although these definitions appear quite different from the classical ideas of differential equations and solutions, a relationship may be established by supposing the fibred submanifold to be defined by a bundle map, as follows. Let (F, ρ, M) be a second bundle, and let $f : J^k\pi \longrightarrow F$ be a bundle map of constant rank over the identity on M. Any such map f gives rise to a map $\mathcal{D}_f : \Gamma_{loc}(\pi) \longrightarrow \Gamma_{loc}(\rho)$ by the rule $(\mathcal{D}_f\phi)(p) = f(j_p^k\phi)$; $\mathcal{D}_f$ is called a *differential operator* from π to ρ. Now let χ be a fixed section of ρ; ρ might be a trivial vector bundle and χ its zero section, but this need not necessarily be the case. Define $R_{f,\chi} \subset J^k\pi$ by

$$R_{f,\chi} = \{j_p^k\phi : f(j_p^k\phi) = \chi(p)\}.$$

If $\pi_k|_{R_{f,\chi}}$ is a fibred submanifold of π_k then it is called the *differential equation determined by the differential operator $\mathcal{D}_f$ and the section χ.*

An example:

- Let π and ρ be copies of the trivial bundle $(\mathbf{R} \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates (x, u). Let f be the bundle map from π_1 to ρ defined by

  $$u(f(j_p^1\phi)) = (u_x - u)(j_p^1\phi)$$

  and let χ be the section of ρ defined by $\chi(x) = (x, e^x)$. Then $R_{f,\chi}$ is defined by the equation

  $$u_x - u = u \circ \chi.$$

  One solution of $R_{f,\chi}$ is the section of π defined by $\phi(x) = (x, xe^x)$. In fact, any solution of this equation must satisfy the classical equation

  $$\phi'(x) - \phi(x) = e^x.$$

Finally I should mention that my definition of a differential equation is not the most general one possible. To see this, let π and ρ be the trivial bundles of the last example but let f now be the bundle map defined by

$$u(f(j^1_p\phi)) = (uu_x - x)(j^1_p\phi)$$

and let χ be the zero section of ρ. Then $R_{f,\chi}$ is a submanifold of $J^1\pi$ which is not fibred over the base manifold $\mathbf{R}$ because the inverse image of zero is not a manifold (it is homeomorphic to the cartesian axes in $\mathbf{R}^2$). In this case the map f does not have constant rank. It would be possible to extend the definition of a differential equation to allow examples of this nature by considering arbitrary submanifolds of jet manifolds, but one would also need "solutions" which were multi-valued or had "infinite derivatives" [55]: in the present case the solutions are described by concentric circles in E. Consideration of these questions would, however, take me too far afield.

2.2 Infinite Jets

A number of results in the theory of jets can be formulated more clearly by eliminating the need for counting the order of the jet manifold on which the construction is defined. This can be done by using "infinite jets". There are two approaches to this idea. One is to regard the "infinite jet manifold" as merely a convenient fiction, and to regard entities defined on different jet manifolds as equivalent when they are related by the appropriate projection maps; these equivalence classes are then the corresponding entities defined on the fictitious manifold "$J^\infty\pi$". This approach is adopted by Kuperschmidt [44] and Tulczyjew [67], whose work I cite elsewhere in this dissertation. With this approach, one has to keep in mind just which properties the various entities are meant to possess: for example, a "vector field" on "$J^\infty\pi$" is actually an equivalence class of vector fields, and there is no reason *a priori* why such an object should have any of the standard properties of vector fields.

I shall adopt the alternative approach, which is to define $J^\infty\pi$ as a *bona fide* manifold. To do this, I shall use the ideas of inverse and direct limits from category theory: see, for example, [45]. The present section will therefore have a rather different flavour from the rest of this dissertation.

Definition 2.2.1 *Let O_k ($k \in \mathbf{N}$) be a family of objects in a category, and let $f_{k+1,k} : O_{k+1} \longrightarrow O_k$ be a family of morphisms. The* inverse limit *of the family $(O_k, f_{k+1,k})$—if it exists—is the object O_∞ and the family of morphisms $f_{\infty,k} : O_\infty \longrightarrow O_k$ satisfying the following two properties:*

1. *for each k, $f_{\infty,k} = f_{k+1,k} \circ f_{\infty,k+1}$;*

2. *if O is another object and $g_k : O \longrightarrow O_k$ another family of morphisms satisfying the first property then there is a unique morphism $g : O \longrightarrow O_\infty$ such that, for each k, $g_k = f_{\infty,k} \circ g$.*

If all the arrows mentioned in this definition point the other way then O_∞ is the direct limit *of* $(O_k, f_{k,k+1})$.

This definition is summarised by the following two diagrams.

$$\begin{tikzcd}
 & & O \arrow[d, "g"] \arrow[dl, "g_{k+1}"'] \arrow[dr, "g_k"] & & \\
 & & O_\infty \arrow[dl, "f_{\infty,k+1}"] \arrow[dr, "f_{\infty,k}"'] & & \\
 {} \arrow[r] & O_{k+1} \arrow[rr, "f_{k+1,k}"'] & & O_k \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & O_k \arrow[rr, "f_{k,k+1}"] \arrow[dr, "f_{k,\infty}"] \arrow[ddr, "g_k"'] & & O_{k+1} \arrow[r] \arrow[dl, "f_{k+1,\infty}"'] \arrow[ddl, "g_{k+1}"] & {} \\
& & O_\infty \arrow[d, "g"'] & & \\
& & O & &
\end{tikzcd}$$

Note that these limits (if they exist) are necessarily unique to within an isomorphism in the appropriate category.

The definition of inverse limit is sometimes applied directly in the category of differentiable manifolds and smooth maps to the family $(J^k\pi, \pi_{k+1,k})$. However, this leaves unanswered the questions of whether such an inverse limit exists in this category, and (if it does) whether the object constructed has the properties one would expect of an infinite jet manifold; it also gives no indication of the nature of the vector space on which the manifold is modelled. I shall therefore apply these definitions in the category *TVS* of topological vector spaces and continuous linear maps.

Consider first the family of objects $\mathbf{R}^n$, and maps $p_{n+1,n} : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n$ given by projection on the first n coordinates.

Lemma 2.2.2 *The family* $(\mathbf{R}^n, p_{n+1,n})$ *has an inverse limit in the category TVS.*

Proof Let $\mathbf{R}^\infty$ be the set of all infinite sequences of real numbers, with vector space structure given by coordinate operations. Let $p_{\infty,n} : \mathbf{R}^\infty \longrightarrow \mathbf{R}^n$ be projection on the first n coordinates. Let $\mathbf{R}^\infty$ have the topology given by letting subsets of the form $p^{-1}_{\infty,n}(O_n)$, $O_n \subset \mathbf{R}^n$, O_n open, be a basis for the open sets. It may be checked that $\mathbf{R}^\infty$ becomes a topological vector space, that the maps $p_{\infty,n}$ are linear and continuous, and that $\mathbf{R}^\infty$ is the inverse limit of $(\mathbf{R}^n, p_{n+1,n})$. ∎

The dual result concerns the family of objects $\mathbf{R}^{n*}$ (which for the moment I distinguish conceptually from $\mathbf{R}^n$) and maps $i_{n,n+1} : \mathbf{R}^{n*} \longrightarrow \mathbf{R}^{n+1*}$ given by isomorphism with the subspace of points whose last coordinate is zero (so that $i_{n,n+1}$ is the transpose of $p_{n+1,n}$).

Lemma 2.2.3 *The family* $(\mathbf{R}^{n*}, i_{n,n+1})$ *has a direct limit in the category TVS.*

Proof Let $\mathbf{R}_0^\infty$ be the set of all infinite sequences of real numbers with only finitely-many non-zero coordinates, with vector space structure given by coordinate operations. Let $i_{n,\infty} : \mathbf{R}^{n*} \longrightarrow \mathbf{R}_0^\infty$ be the isomorphism with the subspace of points whose m-th coordinates $(m > n)$ are zero. Let $\mathbf{R}_0^\infty$ have the topology given by specifying that $O \subset \mathbf{R}_0^\infty$ is open if, and only if, for each $n \in \mathbf{N}$, $i_{n,\infty}^{-1}(O)$ is open in $\mathbf{R}^{n*}$. It may be checked that $\mathbf{R}_0^\infty$ becomes a topological vector space, that the maps $i_{n,\infty}$ are linear and continuous, and that $\mathbf{R}_0^\infty$ is the direct limit of $(\mathbf{R}^{n*}, i_{n,n+1})$. ∎

It is well-known that the study of infinite-dimensional spaces (and manifolds) contains many pitfalls for the unwary. Happily, however, the following result is true.

Proposition 2.2.4 *The topological dual of* $\mathbf{R}^\infty$ *is isomorphic to the vector space* $\mathbf{R}_0^\infty$*; the topological dual of* $\mathbf{R}_0^\infty$ *is isomorphic to the vector space* $\mathbf{R}^\infty$.

Proof Suppose $\alpha : \mathbf{R}^\infty \longrightarrow \mathbf{R}$ is linear. If α depends only on finitely many components of its argument—that is, if there is a natural number n such that if $x, y \in \mathbf{R}^\infty$ and $x_k = y_k$ for $k \leq n$ then $\alpha(x) = \alpha(y)$—then α is continuous. For define $\alpha_n : \mathbf{R}^n \longrightarrow \mathbf{R}$ by $\alpha_n = \alpha \circ i_{n,\infty}$, where now $i_{n,\infty} : \mathbf{R}^n \longrightarrow \mathbf{R}^\infty$. Then the condition on α implies that $\alpha = \alpha_n \circ p_{\infty,n}$ where $p_{\infty,n} : \mathbf{R}^\infty \longrightarrow \mathbf{R}^n$, and since α_n and $p_{\infty,n}$ are continuous, so is α. Hence $\alpha \in \mathbf{R}^{\infty*}$. Conversely, suppose α depends on infinitely many components of its argument. Then for every $n \in \mathbf{N}$ there are $x_{(n)}, y_{(n)} \in \mathbf{R}^\infty$ with $x_{(n)k} = y_{(n)k}$ for $k \leq n$ but $\alpha(x_{(n)}) \neq \alpha(y_{(n)})$. Consider the sequence of elements $z_{(n)} \in \mathbf{R}^\infty$ defined by

$$z_{(n)} = \frac{x_{(n)} - y_{(n)}}{\alpha(x_{(n)} - y_{(n)})}.$$

Then $z_{(n)} \longrightarrow 0 \in \mathbf{R}^\infty$ as $n \longrightarrow \infty$: for let $O \subset \mathbf{R}^\infty$ be open with $0 \in O$, and write

$$O = \bigcup_\lambda p_{\infty,n_\lambda}^{-1}(O_\lambda) \qquad O_\lambda \subset \mathbf{R}^n, \quad O_\lambda \text{ open.}$$

Choose μ with $O \in p_{\infty,n_\mu}^{-1}(O_\mu)$; then if $n > n_\mu$ the first n_μ coordinates of $x_{(n)}$ are zero and so $z_{(n)} \in p_{\infty,n_\mu}^{-1}(O_\mu) \subset O$. However, $\alpha(z_{(n)}) = 1$ for each

n, so α is not continuous. Consequently the map $P : \mathbf{R}_0^\infty \longrightarrow \mathbf{R}^{\infty *}$ defined by, for $y \in \mathbf{R}_0^\infty$ and $x \in \mathbf{R}^\infty$,

$$Py(x) = \sum_{k=1}^{\infty} y_k x_k$$

(finite sum since $y \in \mathbf{R}_0^\infty$) is a canonical linear isomorphism.

To prove the second part of the proposition, note that $\mathbf{R}^\infty$ may be identified with the algebraic dual of $\mathbf{R}_0^\infty$: for given $x \in \mathbf{R}^\infty$, the map $Qx : \mathbf{R}_0^\infty \longrightarrow \mathbf{R}$ defined by

$$Qx(y) = \sum_{k=1}^{\infty} x_k y_k$$

(finite sum since $y \in \mathbf{R}_0^\infty$) is obviously linear; conversely, given any linear map $\alpha : \mathbf{R}_0^\infty \longrightarrow \mathbf{R}$, put $\alpha_i = \alpha(e_{(i)})$ where $e_{(i)k} = \delta_{ik}$, and define $x_{(\alpha)} \in \mathbf{R}^\infty$ by $x_{(\alpha)i} = \alpha_i$. Then $Qx_{(\alpha)} = \alpha$ because the $e_{(i)}$ form a basis of $\mathbf{R}_0^\infty$. I can also show that each linear map $\mathbf{R}_0^\infty \longrightarrow \mathbf{R}$ is continuous; for if α is such a map, then for each $n \in \mathbf{N}^+$, $\alpha \circ i_{n,\infty} : \mathbf{R}^{n*} \longrightarrow \mathbf{R}$ is linear and hence continuous. So if $O \subset \mathbf{R}$ is open then for each $n \in \mathbf{N}^+$, $i_{n,\infty}^{-1}(\alpha^{-1}(O)) = (\alpha \circ i_{n,\infty})^{-1}(O)$ is open in $\mathbf{R}^{n*}$; consequently $\alpha^{-1}(O)$ is open in $\mathbf{R}_0^\infty$. ∎

I shall be more interested in $\mathbf{R}^\infty$ than $\mathbf{R}_0^\infty$, since the former will be the model space for $J^\infty\pi$. There is now a reasonable literature on Banach manifolds; however, it turns out the topology on $\mathbf{R}^\infty$ is *not* generated by a norm. This fact has several consequences. One of the most relevant for my purposes is that a vector field defined on $\mathbf{R}^\infty$ need not have a flow, even locally: the usual proof of the existence of flows requires that the vector field satisfy a Lipschitz condition which is meaningless in the absence of a norm.

Proposition 2.2.5 $\mathbf{R}^\infty$ *is a Fréchet space but not a Banach space.*

Proof A Fréchet space is a complete, locally convex, metrizable topological vector space; it is straightforward to check that $\mathbf{R}^\infty$ has these properties (see, for example, [66]).

To see that $\mathbf{R}^\infty$ is not a Banach space, suppose that there were a norm defining the topology. Let $e_{(i)} \in \mathbf{R}^\infty$ be defined by $e_{(i)j} = \delta_{ij} \in \mathbf{R}$ and let $x, x_{(n)} \in \mathbf{R}^\infty$ $(n \in \mathbf{N}^+)$ be defined by

$$\begin{aligned} x_j &= \frac{3^j}{\|e_{(j)}\|} \\ x_{(n)j} &= x_j \qquad (j \leq n) \\ &= 0 \qquad (j > n) \end{aligned}$$

Then $x_{(n)} \longrightarrow x$ in the topology of $\mathbf{R}^\infty$ (using an argument similar to that used in the first part of Proposition 2.2.4) and so $\|x_{(n)}\| \longrightarrow \|x\|$. However, the triangle inequality shows that

$$\|x_{(n)}\| \geq \tfrac{3}{2}(3^n + 1)$$

so that $\|x_{(n)}\| \longrightarrow \infty$. Therefore such a norm cannot exist. ∎

The following Lemma (whose proof I omit) summarises some properties of $\mathbf{R}^\infty$.

Lemma 2.2.6 *$\mathbf{R}^\infty$ is a second-countable, path-connected Fréchet space. Furthermore, for any $n \in \mathbf{N}$, $\mathbf{R}^n \times \mathbf{R}^\infty \cong \mathbf{R}^\infty$.*

∎

I now recall (from, for example, [46]) the definition of differentiability for functions defined on open subsets of a topological vector space.

Definition 2.2.7 *Let E, F be Fréchet spaces, U a neighbourhood of $0 \in E$, and $f : U \longrightarrow F$. Then f is said to be* tangent to zero *if, for every neighbourhood W of $0 \in F$, there is a neighbourhood V of $0 \in E$ and a function $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ satisfying*

$$f(tV) \subset |\phi(t)|W \qquad \text{for all } t \in (0, 1)$$

and

$$\lim_{t \to 0} \frac{\phi(t)}{t} = 0.$$

Definition 2.2.8 *Let E, F be Fréchet spaces; let $U \subset E$ be open with $x \in U$. Let $f : U \longrightarrow F$; f is said to be* differentiable at x *if there is a continuous linear map $Df_x : E \longrightarrow F$ such that the map $h \longmapsto f(x+h) - f(x) - Df_x(h)$ is tangent to zero.*

In distinction to the finite-dimensional case, if $f : E \longrightarrow F$ is linear and *continuous* then f is differentiable everywhere and, for each $x \in E$, $Df_x = f$. In particular, this applies to the projection maps $p_{\infty,n} : \mathbf{R}^\infty \longrightarrow \mathbf{R}^n$. The composition of two differentiable maps is differentiable, and the chain rule applies.

Lemma 2.2.9 *Let U be a neighbourhood of $x \in \mathbf{R}^\infty$, and let $f : U \longrightarrow \mathbf{R}^\infty$. Let f_k equal $pr_k \circ f$, where pr_k is projection on the k^{th} factor, and suppose that, for each k, f_k only depends on a finite number n_k of components of its argument—that is, there is a map $f_k' : U_k \longrightarrow \mathbf{R}$ (where U_k is a neighbourhood of $p_{\infty,n_k}(x)$) satisfying $f_k = f_k' \circ p_{\infty,n_k}$. Then if each f_k' is differentiable at p_{∞,n_k}, the map f is differentiable at x.*

Proof Each f_k is differentiable at x, so write F_k for $D(f_k)_x$; then each F_k is a continuous linear map from $\mathbf{R}^\infty$ to $\mathbf{R}$. Define the map $F : \mathbf{R}^\infty \longrightarrow \mathbf{R}^\infty$ by $(F(h))_k = F_k(h)$. Clearly F is linear. The following argument shows that it is continuous. For any open set $O \subset \mathbf{R}^\infty$ of the form $O = p_{\infty,n}^{-1}(O')$ where $O' \subset \mathbf{R}^n$ is open,

$$\begin{aligned}
F^{-1}(O) &= F^{-1}(p_{\infty,n}^{-1}(O')) \\
&= (p_{\infty,n} \circ F)^{-1}(O') \\
&= (p_{\infty,n} \circ (F_1 \times F_2 \times \ldots))^{-1}(O') \\
&= (F_1 \times \ldots \times F_n)^{-1}(O') \\
&= (F_1 \times \ldots \times F_n)^{-1}\left(\bigcup_\lambda (O_1^{(\lambda)} \times \ldots \times O_n^{(\lambda)})\right) \quad O_i^{(\lambda)} \subset \mathbf{R} \text{ open} \\
&= \bigcup_\lambda (F_1 \times \ldots \times F_n)^{-1}(O_1^{(\lambda)} \times \ldots \times O_n^{(\lambda)}) \\
&= \bigcup_\lambda \bigcap_{i=1}^n F_i^{-1}(O_i^{(\lambda)})
\end{aligned}$$

which is open in $\mathbf{R}^\infty$ by continuity of each F_i. F is then clearly the derivative of f at x. ∎

I shall now use this machinery to define the infinite jet manifold. As elsewhere, (E, π, M) is a given bundle.

Definition 2.2.10 *If $\phi \in \Gamma_{loc}(\pi)$ and $p \in$ domain(ϕ) then the* infinite jet of ϕ at p *is defined to be*

$$j_p^\infty \phi = \bigcap_{k=0}^\infty j_p^k \phi.$$

This slightly obscure definition just means that two local sections are equivalent at p if their values and *all* their derivatives are equal at p, and the infinite jet is just the equivalence class. A couple of technicalities are that $j_p^\infty \phi$ is never empty (for ϕ is a member of it) and that if $p \neq q$ then $j_p^\infty \phi$ never equals $j_q^\infty \psi$ for any $\phi, \psi \in \Gamma_{loc}(\pi)$. The following definition therefore makes sense.

Definition 2.2.11 *The set of all infinite jets of local sections of π, at all points of M, is called the* infinite jet manifold of π, *and denoted $J^\infty \pi$:*

$$J^\infty \pi = \{ j_p^\infty \phi : \phi \in \Gamma_{loc}(\pi), p \in \text{domain}(\phi) \}.$$

Proposition 2.2.12 *$J^\infty \pi$ actually is an infinite-dimensional manifold.*

Proof Just as in Proposition 2.1.6, define a set $U^\infty \subset J^\infty\pi$ and a coordinate map $u^\infty : U^\infty \longrightarrow \mathbf{R}^\infty$ given by

$$u^\infty(j^\infty_q\psi) = \left(x^i(q), \psi^\alpha(q), \left.\frac{\partial^{|I|}\psi^\alpha}{\partial x^I}\right|_q \right), \qquad 1 \leq |I| < \infty.$$

Note that, in distinction to the case with finite jets, the fact that $u^\infty(U^\infty)$ is open in $\mathbf{R}^\infty$ is non-trivial: it depends on the fact that an arbitrary family of Taylor coefficients defines a C^∞ (though not necessarily analytic) real-valued function in a neighbourhood of $0 \in \mathbf{R}^m$: see [38]. In fact, by using Lemma 2.2.6, one finds that $u^\infty(U^\infty) \cong W \times \mathbf{R}^\infty$ where W is the image of U under the coordinate system (x^i, u^α). As before, I shall denote the real-valued function

$$j^\infty_q\psi \longmapsto \left.\frac{\partial^{|I|}\psi^\alpha}{\partial x^I}\right|_q$$

by u^α_I. I shall now show that the transition functions are C^∞. So suppose that v^∞ is another coordinate map. Then by an argument similar to that in Proposition 2.1.6, one sees that each v^β_J does not depend on u^α_I with $|I| > |J|$. Consequently each $v^\beta_J \circ (u^\infty)^{-1}$ is a map from $\mathbf{R}^\infty \longrightarrow \mathbf{R}$ which depends only on a finite number of the components of its argument, and the corresponding map $\mathbf{R}^N \longrightarrow \mathbf{R}$ (N finite) is differentiable (actually C^∞). Then Lemma 2.2.9 shows that $v^\infty \circ (u^\infty)^{-1}$ is differentiable at each point; applying the argument recursively shows that $v^\infty \circ (u^\infty)^{-1}$ is C^∞. Again one may check that the topology induced on $J^\infty\pi$ is Hausdorff, second-countable and connected; consequently $J^\infty\pi$ is an infinite-dimensional manifold. ∎

I can define maps linking $J^\infty\pi$ to the finite jet manifolds.

Definition 2.2.13 *For $k \in \mathbf{N}$, the map $\pi_{\infty,k} : J^\infty\pi \longrightarrow J^k\pi$ is defined by $\pi_{\infty,k}(j^\infty_p\phi) = j^k_p\phi$; the map $\pi_\infty : J^\infty\pi \longrightarrow M$ is defined by $\pi_\infty(j^\infty_p\phi) = p$.*

Lemma 2.2.14 *$\pi_{\infty,k}$ and π_∞ are bundles.*

∎

Lemma 2.2.15 *$(J^\infty\pi, \pi_{\infty,k})$ is the inverse limit of $(J^k\pi, \pi_{k+1,k})$ in the category of (arbitrary-dimensional) differentiable manifolds and smooth maps.*

Proof $(J^\infty\pi, \pi_{\infty,k})$ clearly satisfies the first property of Definition 2.2.1. If (O, g_k) is another manifold and family of maps satisfying that property then define $g : O \longrightarrow J^\infty\pi$ by, for each $x \in O$, putting $p = \pi(g_0(x))$ and letting $\phi \in \Gamma_{loc}(\pi)$ have domain including p and have derivative coordinates at p

corresponding to the coordinates of the various $g_k(x) \in J^k\pi$: these coordinates are compatible by the properties of the g_k, and such a ϕ exists for the same reason as that given in the proof of Proposition 2.2.12. Consequently $g(x)$ may be defined to equal $j_p^\infty \phi$. ∎

As in the case of finite jet manifolds, the subset $J_p^\infty \phi \subset J^\infty \pi$ of ∞-jets of local sections at a particular point $p \in M$ is an embedded submanifold.

An example:

- If π is the trivial bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$ then I shall denote $J_0^\infty \pi$ instead by $T^\infty F$ and refer to it as the *infinite-order tangent manifold of* F. Note particularly that the elements of $T^\infty F$ may be considered as equivalence classes of C^∞ (rather than analytic) curves in F; this point will be relevant in Chapter 7 when I shall consider $T^\infty G$ where G is a Lie group (and therefore an analytic manifold).

My final remarks in this Section are about the tangent and cotangent spaces to $J^\infty \pi$ at a point $j_p^\infty \phi$, and the corresponding spaces of vector fields and differential forms. Observe first that defining a tangent vector as an equivalence class of curves gives a tangent space $T_{j_p^\infty \phi}(J^\infty \pi)$ isomorphic to $\mathbf{R}^\infty$; one may express any tangent vector ξ as a "formal infinite sum"

$$\xi = \xi^i \left. \frac{\partial}{\partial x^i} \right|_{j_p^\infty \phi} + \sum_{|I|=0}^{\infty} \xi_I^\alpha \left. \frac{\partial}{\partial u_I^\alpha} \right|_{j_p^\infty \phi} .$$

No question of convergence arises at this stage, because the "sum" is just a coordinate representation as an infinite sequence. There is potential for a problem to arise when ξ acts as a derivation on a function f defined in a neighbourhood of $j_p^\infty \phi$; however, ξf always turns out to be a finite sum of real numbers. This is because the cotangent space $T^*_{j_p^\infty \phi}(J^\infty \pi)$ is defined to be the topological dual of the tangent space $T_{j_p^\infty \phi}(J^\infty \pi)$ and hence is isomorphic to $\mathbf{R}_0^\infty$; therefore there is a $k \in \mathbf{N}$ such that

$$df|_{j_p^\infty \phi} = \left. \frac{\partial F}{\partial x^i} \right|_{j_p^\infty \phi} dx^i \Big|_{j_p^\infty \phi} + \sum_{|I|=0}^{k} \left. \frac{\partial f}{\partial u_I^\alpha} \right|_{j_p^\infty \phi} du_I^\alpha|_{j_p^\infty \phi}$$

and $\left. \frac{\partial f}{\partial u_I^\alpha} \right|_{j_p^\infty \phi} = 0$ for $|I| > k$.

In a similar way, a vector field X on $J^\infty \pi$ may be written in coordinates as a formal infinite sum

$$X = X^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} X_I^\alpha \frac{\partial}{\partial u_I^\alpha} .$$

However, although a cotangent vector only has a finite number of terms in its coordinate expansion, there is no reason why this should be true globally for a differential form.

A simple example:

- Let π be the trivial bundle $(\mathbf{R} \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates (x, u) on $\mathbf{R} \times \mathbf{R}$ and $(x, u_{(k)})$ on $J^\infty\pi$. Let $b : \mathbf{R} \longrightarrow \mathbf{R}$ be a C^∞ bump function satisfying $b(0) = 1$, $b(t) = 0$ for $t \notin (-\frac{1}{2}, \frac{1}{2})$. Let ω be the differential form given in coordinates as

$$\omega\big|_{j_p^\infty\phi} = \sum_{k=0}^{\infty} b(x(p) - k)du_{(k)}\Big|_{j_p^\infty\phi}$$

which at any given point of $J^\infty\pi$ just consists of a single term but globally cannot be expressed as a finite sum of terms involving the forms $du_{(k)}$.

In fact this property isn't even true locally, although to demonstrate this I shall look first at differentiable functions defined on $\mathbf{R}^\infty$. The idea is to take a smooth function which depends on infinitely many components, constructed in a similar way to the differential form of the previous example. By a suitable transformation, the component dependence can be made to accumulate in a neighbourhood of the origin. The resulting function may not be smooth; however, multiplying it by a smooth function whose derivatives all vanish at the origin gives a new function with the required properties. The following construction gives an explicit construction for a function which is formed in essentially this way.

- There is a smooth function $f : \mathbf{R}^\infty \longrightarrow \mathbf{R}$ such that in any neighbourhood of the origin f depends on infinitely many components. For let $b : \mathbf{R} \longrightarrow \mathbf{R}$ be a bump function as before, and let

$$f(x) = \sum_{k=2}^{\infty} 2^{-2^k} b(2^k x_1 - 1) \sin x_k.$$

At each $x \in \mathbf{R}^\infty$ the sum only contains a finite number of terms, and in fact each $x \neq 0$ has a neighbourhood in which there are only a finite number of terms, so that f is certainly smooth at those points. To see that f is differentiable at the origin (with derivative zero) note that $h \longmapsto f(h)$ is tangent to zero: for given a neighbourhood $(-a, a)$ of zero in $\mathbf{R}$, choose $V \cong (-1, 1) \times \mathbf{R}^\infty$ open in $\mathbf{R}^\infty$ and $\phi(t) = \frac{1}{a}2^{-\frac{1}{2t}}$ in Definition 2.2.7. So if $h \in tV$ then $|h_1| < t$ and the only terms which contribute to the value of f are those with $2^{-k} < 2t$, giving

$$f(h) \in [0, 2^{-\frac{1}{2t}}) \subset |\phi(t)|(-a, a).$$

The argument may be applied recursively to show that f is smooth at the origin.

One may use this as a model for a similar function on $J^\infty\pi$, and hence (by taking its differential) as a model for a 1-form which depends on infinitely many components. In terms of my previous example, this particular 1-form would appear in coordinates as

$$\begin{aligned}\omega|_{j_p^\infty\phi} &= \left(\sum_{k=0}^{\infty} 2^{(k-2^k)}b'(2^k x(p)-1)\sin\left.\frac{\partial^k\phi}{\partial x^k}\right|_p\right) dx|_{j_p^\infty\phi} \\ &\quad + \sum_{k=0}^{\infty}\left(2^{-2^k}b(2^k x(p)-1)\cos\left.\frac{\partial^k\phi}{\partial x^k}\right|_p du_{(k)}\Big|_{j_p^\infty\phi}\right).\end{aligned}$$

Consequently, my spaces $C^\infty(J^\infty\pi)$ and $\bigwedge^r J^\infty\pi$ are larger than the corresponding spaces used by those authors (for example [56], [67]) who take direct limits of the corresponding spaces on finite jet manifolds rather than construct an infinite-dimensional manifold. In fact, the most I can say about my space $C^\infty(J^\infty\pi)$ is a consequence of the following lemma about differentiable functions on $\mathbf{R}^\infty$.

Lemma 2.2.16 *If $U\subset\mathbf{R}^\infty$ is open and $f:U\longrightarrow\mathbf{R}$ is differentiable then for each $x\in U$ there is $n_x\in\mathbf{N}^+$ such that if $y\in U$ and $p_{\infty,n_x}(y)=p_{\infty,n_x}(x)$ then $f(y)=f(x)$.*

Proof Suppose first that $x=0$ and that $f(x)=0$, $Df_x(h)=0$. Then $h\longmapsto f(h)$ is tangent to zero, so given the open interval $(-a,a)$ I may certainly find a neighbourhood V of $0\in\mathbf{R}^\infty$ satisfying $f(tV)\subset|t|(-a,a)$ for $t\in(0,1)$; I may also take V to be of the form $I_1\times\ldots\times I_n\times\mathbf{R}^\infty$, where I_i are open intervals.

Now suppose $y\in U$ and $p_{\infty,n}(y)=p_{\infty,n}(0)$; then $y\in V$ and indeed $y\in tV$ for $t\in(0,1)$. Therefore $f(y)\in(-ta,ta)$ for each t, and consequently $f(y)=0$. Since differentiability is translation invariant, the argument is actually independent of the conditions on x and $f(x)$; furthermore, Df_x is a continuous linear functional and so depends only on a finite number of components, and one may therefore apply the same argument to the function $f-Df_x$. ∎

Corollary 2.2.17 *If $f\in C^\infty(J^\infty\pi)$ then for each $j_p^\infty\phi\in J^\infty\pi$ there is $n\in\mathbf{N}$ such that*

$$\left.\frac{\partial f}{\partial u_I^\alpha}\right|_{j_p^\infty\phi}=0 \qquad \textit{for } |I|>n.$$

Proof Either from the preceding lemma, or directly from the properties of the cotangent space to $J^\infty\pi$ at $j_p^\infty\phi$. ∎

2.3 Total Derivatives and Contact Forms

In Chapter 1 I explained how, on an arbitrary bundle (E, π, M), there was a natural description of vertical tangent vectors (and vector fields) and of horizontal cotangent vectors (and differential forms). I also pointed out that the complementary entities to these were actually equivalence classes, and that the choice of a distinguished representative of each equivalence class was the same as specifying a *connection* on the bundle. More precisely, a connection would be defined as a distribution Δ on E such that, at each point $a \in E$, $V_a\pi \oplus \Delta_a = T_aE$; the other properties would follow imediately from this definition. (In certain circumstances one would also require a connection to have additional properties, such as horizontal completeness or equivariance with respect to a group action.) When studied in the context of jet bundles, each such connection is seen to fall naturally into two parts. One part is a skeleton which is common to all connections on that bundle; the other is a function from E to its first jet manifold $J^1\pi$ which gives each connection its individual properties. I shall elaborate on this decomposition in Chapter 4; in the present section I shall describe the bones which build up the skeleton. These bones comprise the *total derivatives* (which, when flesh is added, become the horizontal vector fields associated to the connection) and the *contact forms* (which become the vertical 1-forms).

Essentially, a total derivative of order k is a particular type of vector field along the map $\pi_{k+1,k}$ which is generated by vector fields lifted from M in a natural way; each vector field along $\pi_{k+1,k}$ may be written uniquely as the sum of a total derivative of order k and another vector field along $\pi_{k+1,k}$ which is vertical over M. The construction is carried out pointwise, so I shall give a description in terms of bundles.

Definition 2.3.1 *The k^{th}* holonomic lift *of tangent vectors on M is the map from the bundle $J^{k+1}\pi \times_{J^k\pi} \pi_k^*TM \longrightarrow J^k\pi$ to the tangent bundle $\tau_{J^k\pi} : J^k\pi \longrightarrow TJ^k\pi$ defined by*

$$(j_p^{k+1}\phi, (j_p^k\phi, \xi)) \longmapsto (j^k\phi)_*\xi \qquad k \in \mathbf{N}.$$

I must check that this is well-defined for different choices of ϕ with the same k-jet at p. So suppose $\gamma : \mathbf{R} \longrightarrow M$ is a curve with $\gamma(0) = p$, $[\gamma] = \xi$. Then $(j^k\phi)_*\xi = [j^k\phi \circ \gamma]$ and

$$\left.\frac{d}{dt}\right|_{t=0} u_I^\alpha \circ j^k\phi \circ \gamma = \left.\frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}}\right|_p \left.\frac{d\gamma^i}{dt}\right|_{t=0}$$

depends only on the derivatives of ϕ of order $\leq k+1$ so is well-defined at $(j_p^{k+1}\phi, (j_p^k\phi, \xi))$. I may therefore use this construction to lift a vector field on M to a vector field along $\pi_{k+1,k}$.

Definition 2.3.2 *If $X \in \mathcal{X}(M)$, the k^{th}* holonomic lift *of X is the vector field $X^k \in \mathcal{X}(\pi_{k+1,k})$ defined by*

$$X^k_{j_p^{k+1}\phi} = (j^k\phi)_* X_p \qquad k \in \mathbf{N}.$$

A similar pair of definitions applies when $k = \infty$; however, the holonomic lift of tangent vectors has a simpler expression and the holonomic lift of a vector field on M is a vector field on a manifold rather than along a map.

Definition 2.3.3 *The* infinite holonomic lift *of tangent vectors on M is the map from the bundle $\pi_\infty^*(\tau_M)$ to $\tau_{J^\infty\pi}$ defined by*

$$(j_p^\infty\phi, \xi) \longmapsto (j^\infty\phi)_*\xi.$$

Definition 2.3.4 *If $X \in \mathcal{X}(M)$, the* infinite holonomic lift of X *is the vector field $X^\infty \in \mathcal{X}(J^\infty\pi)$ defined by*

$$X^\infty_{j_p^\infty\phi} = (j^\infty\phi)_* X_p.$$

In fact, by adopting the convention that $\pi_{k+1,k}$ is the identity map on $J^\infty\pi$ when $k = \infty$ and making an appropriate identification of bundles, both sets of definitions may be expressed by the same formulæ. I shall normally do this in future without additional comment, except where it seems important to point out the difference between a vector field on a manifold and one merely along a map.

Example:

- If π is the trivial bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$ then the holonomic lift of the vector field $\frac{\partial}{\partial t}$ on $\mathbf{R}$ is the "total time derivative" operator $\frac{d}{dt}$. (A more detailed description of this will be given shortly).

A characterisation of X^k as a Lie derivative acting on functions may be found by taking $f \in C^\infty(J^k\pi)$ and unwinding the definitions.

$$\begin{aligned}
(d_{X^k} f)(j_p^{k+1}\phi) &= X^k_{j_p^{k+1}\phi}(f) \\
&= (j^k\phi)_* X_p(f) \\
&= X_p(f \circ j^k\phi) \\
&= d_X(f \circ j^k\phi)(p).
\end{aligned}$$

Another way of writing this is

$$(d_{X^k} f) \circ j^{k+1}\phi(p) = d_X(f \circ j^k\phi)(p)$$

which gives

$$(j^{k+1}\phi)^* \circ d_{X^k}(f) = d_X \circ (j^k\phi)^*(f)$$

and therefore

$$(j^{k+1}\phi)^* \circ d_{X^k} = d_X \circ (j^k\phi)^*$$

for every $\phi \in \Gamma_{loc}(\pi)$. An obvious corollary is $d_{X^k} \circ \pi_k^* = \pi_{k+1}^* \circ d_X$.

Of course, all this applies equally to vector fields defined only locally on M, for example the coordinate vector field $\frac{\partial}{\partial x^i}$. The coordinate representation of its holonomic lift may be found as follows. If $f \in C^\infty(J^k\pi)$,

$$\begin{aligned}
\left.\left(\frac{\partial}{\partial x^i}\right)^k\right|_{j_p^{k+1}\phi} f &= \frac{\partial}{\partial x^i}(f \circ j^k\phi)(p) \\
&= \left.\frac{\partial f}{\partial x^i}\right|_{j_p^k\phi} + \sum_{|I|=0}^{k} \left.\frac{\partial f}{\partial u_I^\alpha}\right|_{j_p^k\phi} \left.\frac{\partial}{\partial x^i}\right|_p \frac{\partial^{|I|}\phi^\alpha}{\partial x^I} \\
&= \left.\frac{\partial f}{\partial x^i}\right|_{j_p^k\phi} + \sum_{|I|=0}^{k} \left.\frac{\partial f}{\partial u_I^\alpha}\right|_{j_p^k\phi} u_{I+1_i}^\alpha(j_p^{k+1}\phi)
\end{aligned}$$

so that

$$\left(\frac{\partial}{\partial x^i}\right)^k = \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} u_{I+1_i}^\alpha \frac{\partial}{\partial u_I^\alpha}$$

which is the total derivative of order k and usually written $\frac{d}{dx^i}$ without any indication of the particular value of k. Recall also from the definition of $T(J^\infty\pi)$ in Section 2.2 that a vector field on $J^\infty\pi$ may have infinitely many components, so that this expression still makes sense as a formal sum with $k = \infty$; the derivation of the formula is valid, too, as a consequence of Corollary 2.2.17. In general I shall use the term *k-th order total derivative* to refer, not only to the k-th order holonomic lift X^k of a vector field $X \in \mathcal{X}(M)$, but to any vector field along $\pi_{k+1,k}$ which may be written as a linear combination (by functions on $J^{k+1}\pi$) of holonomic lifts.

An example:

- If π is the trivial bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$ with coordinates (t, q^α) then the coordinate representation of the total time derivative $\frac{d}{dt} \in \mathcal{X}(\pi_{k+1,k})$ is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{r=0}^{k} q_{(r+1)}^\alpha \frac{\partial}{\partial q_{(r)}^\alpha}.$$

Recalling that $J^k\pi \cong \mathbf{R} \times T^kF$, and denoting by $\tau_F^{k+1,k}$ the unique map satisfying

$$\begin{array}{ccc}
\mathbf{R} \times T^{k+1}F & \xrightarrow{pr_2} & T^{k+1}F \\
\Big\downarrow \pi_{k+1,k} & & \Big\downarrow \tau_F^{k+1,k} \\
\mathbf{R} \times T^{k}F & \xrightarrow[pr_2]{} & T^{k}F
\end{array}$$

the total time derivative induces an operator $T \in \mathcal{X}(\tau_F^{k+1,k})$ which is also called a total time derivative operator and which satisfies

$$T_{pr_2(j_p^{k+1}\phi)} = (pr_2)_* \left(\left. \frac{d}{dt} \right|_{j_p^{k+1}\phi} \right).$$

In coordinates,

$$T = \sum_{r=0}^{k} q^\alpha_{(r+1)} \frac{\partial}{\partial q^\alpha_{(r)}}.$$

As usual, the derivation of type d_* corresponding to T will be denoted d_T.

I shall now demonstrate the sense in which the holonomic lifts of vector fields form a Lie algebra. First it is convenient to prove a lemma about characterising vector fields.

Lemma 2.3.5 *Suppose $X, Y \in \mathcal{X}(\pi_{k,l})$. If, for every $\phi \in \Gamma_{loc}(\pi)$, $(j^k\phi)^* \circ d_X = (j^k\phi)^* \circ d_Y$ then $X = Y$.*

Proof This follows directly from the definitions. For every $f \in C^\infty(J^l\pi)$,

$$((j^k\phi)^* \circ d_X)(f) = ((j^k\phi)^* \circ d_Y)(f) \in C^\infty(M)$$

giving, for every $p \in M$,

$$\begin{aligned}
(((j^k\phi)^* \circ d_X)(f))(p) &= (((j^k\phi)^* \circ d_Y)(f))(p) \\
d_X f(j_p^k\phi) &= d_Y f(j_p^k\phi) \\
d_X f &= d_Y f
\end{aligned}$$

since the preceding equality is true for every $j_p^k\phi \in J^k\pi$, and so $X = Y$. ∎

Proposition 2.3.6 *If X,Y are vector fields on M then*

$$\pi^*_{k+2,k+1} \circ d_{[X,Y]^k} = d_{X^{k+1}} \circ d_{Y^k} - d_{Y^{k+1}} \circ d_{X^k}.$$

Proof $d_{X^{k+1}} \circ d_{Y^k} - d_{Y^{k+1}} \circ d_{X^k}$ represents the Lie derivative action of a vector field along $\pi_{k+2,k}$ according to Definition 1.4.8. It satisfies, for any $\phi \in \Gamma_{loc}(\pi)$,

$$\begin{aligned}(j^{k+2}\phi)^* \circ (d_{X^{k+1}} \circ d_{Y^k} - d_{Y^{k+1}} \circ d_{X^k}) &= (d_X \circ d_Y - d_Y \circ d_X) \circ (j^k\phi)^* \\ &= d_{[X,Y]} \circ (j^k\phi)^* \\ &= (j^{k+1}\phi)^* \circ d_{[X,Y]^k} \\ &= (j^{k+2}\phi)^* \circ \pi^*_{k+2,k+1} \circ d_{[X,Y]^k}\end{aligned}$$

using the properties of holonomic lifts and jet extension maps. The result then follows from Lemma 2.3.5. ∎

Consequently $\pi^*_{k+2,k+1} \circ d_{[X,Y]^k} = d_{[X^{k+1},Y^{k+1}]}$ where the bracket on the right-hand side is the Frölicher-Nijenhuis bracket. In particular, the infinitely-prolonged total derivatives do form a *bona-fide* (infinite-dimensional) Lie algebra: $d_{[X,Y]^\infty} = d_{X^\infty} \circ d_{Y^\infty} - d_{Y^\infty} \circ d_{X^\infty}$ so that $[X,Y]^\infty = [X^\infty, Y^\infty]$.

I shall use repeated total derivatives in Section 3.2 when describing the prolongation of vector fields on E. The following version of Leibniz' rule for coordinate total derivatives will therefore be valuable.

Proposition 2.3.7 *If $f, g \in C^\infty(J^k\pi)$ then*

$$\begin{aligned}\frac{d^{|I|}(fg)}{dx^I} &= \sum_{J+K=I} \frac{I!}{J!\,K!}\left(\pi^*_{k+|I|,k+|J|}\frac{d^{|J|}f}{dx^J}\right)\left(\pi^*_{k+|I|,k+|K|}\frac{d^{|K|}g}{dx^K}\right) \\ &\in C^\infty(J^{k+|I|}\pi).\end{aligned}$$

Proof A similar induction to that given in Proposition 2.1.3 but keeping track of the different jet levels on which the functions are defined. ∎

I shall generally omit the pullback maps preceding the various derivatives in the above formula and simply write it as

$$\frac{d^{|I|}(fg)}{dx^I} = \sum_{J+K=I} \frac{I!}{J!\,K!}\frac{d^{|J|}f}{dx^J}\frac{d^{|K|}g}{dx^K}$$

(which of course is literally true if $f, g \in C^\infty(J^\infty\pi)$).

I shall now show how holonomic lifts and total derivatives may be used to construct vertical and horizontal representatives of tangent vectors and vector fields. The most straightforward case is on $J^\infty\pi$, where the holonomic lift does actually define a connection.

Definition 2.3.8 *The* horizontal component *of a tangent vector* ξ *in* $T_{j_p^\infty\phi}(J^\infty\pi)$ *is defined by the composition*

$$\begin{array}{ccccc} T(J^\infty\pi) & \longrightarrow & \pi_\infty^*(TM) & \longrightarrow & T(J^\infty\pi) \\ \xi & \longmapsto & (j_p^\infty\phi, \pi_{\infty *}(\xi)) & \longmapsto & (j^\infty\phi)_*\pi_{\infty *}(\xi) \end{array}$$

and denoted ξ^h. *The* vertical component *of* ξ *is defined by* $\xi^v = \xi - \xi^h$.

It is clear that $\xi^v \in V_{j_p^\infty\phi}\pi_\infty$ and that $(\xi^v)^v = \xi^v$, $(\xi^h)^h = \xi^h$ so that this construction does indeed define a connection. I shall call it the *natural connection* on $J^\infty\pi$. However the equivalent operations cannot define a connection on a finite jet manifold $J^k\pi$ because Definition 2.3.1 requires a $(k+1)$-jet rather than a k-jet; instead, one has the following arrangement.

Definition 2.3.9 *The* horizontal component *of a tangent vector* ξ *in* $T_{j_p^k\phi}(J^k\pi)$ *determined by a point* $j_p^{k+1}\phi \in J^{k+1}\pi$ *is defined by the composition*

$$\begin{array}{ccccc} J^{k+1}\pi \times_{J^k\pi} T(J^k\pi) & \longrightarrow & J^{k+1}\pi \times_{J^k\pi} \pi_k^*TM & \longrightarrow & T(J^k\pi) \\ (j_p^{k+1}\phi, \xi) & \longmapsto & (j_p^{k+1}\phi, (j_p^k\phi, \pi_{k*}\xi)) & \longmapsto & (j^k\phi)_*\pi_{k*}\xi \end{array}$$

and denoted $(j_p^{k+1}\phi, \xi)^h$. *The* vertical component *is defined by* $(j_p^{k+1}\phi, \xi)^v = \xi - (j_p^{k+1}\phi, \xi)^h$.

In later sections I shall also use the notation $h_{j_p^{k+1}\phi}(\xi)$, $v_{j_p^{k+1}\phi}(\xi)$ instead of $(j_p^{k+1}\phi, \xi)^h$, $(j_p^{k+1}\phi, \xi)^v$ (and similarly $h_{j_p^\infty\phi}(\xi)$, $v_{j_p^\infty\phi}(\xi)$ for ξ^h, ξ^v in the case where $\xi \in T(J^\infty\pi)$); h and v will be defined as vector-valued forms along a bundle map.

It is important to note that the operations of taking jet projections and horizontal (or vertical) components commute.

Lemma 2.3.10 *Given* $j_p^{k+1}\phi \in J^{k+1}\pi$ *and* $\xi \in T_{j_p^k\phi}(J^k\pi)$, *if* $0 \leq l < k$ *then*

$$\pi_{k,l*}(j_p^{k+1}\phi, \xi)^h = (j_p^{l+1}\phi, \pi_{k,l*}\xi)^h$$

and similarly for the vertical component.

Proof This follows directly from the definitions of the horizontal and vertical components, and the properties of jet extension maps. For example,

$$\pi_{k,l*}(j_p^{k+1}\phi, \xi)^h = \pi_{k,l*}(j^k\phi)_*\pi_{k*}\xi = (j^l\phi)_*\pi_{l*}\pi_{k,l*}\xi = (j_p^{l+1}\phi, \pi_{k,l*}\xi)^h.$$

∎

I can now apply these constructions to vector fields. There are three different cases to consider.

Definition 2.3.11 *If $X \in \mathcal{X}(J^\infty\pi)$ then the horizontal and vertical components of X are defined to be $X^h, X^v \in \mathcal{X}(J^\infty\pi)$, where*

$$(X^h)_{j_p^\infty\phi} = (X_{j_p^\infty\phi})^h, \quad (X^v)_{j_p^\infty\phi} = (X_{j_p^\infty\phi})^v.$$

Definition 2.3.12 *If $X \in \mathcal{X}(\pi_{k,l})$, $0 \leq l < k \leq \infty$, then the horizontal and vertical components of X are defined to be $X^h, X^v \in \mathcal{X}(\pi_{k,l})$, where*

$$(X^h)_{j_p^k\phi} = (j_p^{l+1}\phi, X_{j_p^k\phi})^h, \quad (X^v)_{j_p^k\phi} = (j_p^{l+1}\phi, X_{j_p^k\phi})^v.$$

Definition 2.3.13 *If $X \in \mathcal{X}(J^k\pi)$, $0 \leq k < \infty$, then the horizontal and vertical components of X are defined to be $X^h, X^v \in \mathcal{X}(\pi_{k+1,k})$, where*

$$(X^h)_{j_p^{k+1}\phi} = (j_p^{k+1}\phi, X_{j_p^k\phi})^h, \quad (X^v)_{j_p^{k+1}\phi} = (j_p^{k+1}\phi, X_{j_p^k\phi})^v.$$

The first two of these definitions yield decompositions of the spaces $\mathcal{X}(J^\infty\pi)$ and $\mathcal{X}(\pi_{k,l})$ (where $k > l$). Writing $\mathcal{X}^h$ and $\mathcal{X}^v$ for the spaces of horizontal and vertical components, one has

$$\begin{aligned} \mathcal{X}(J^\infty\pi) &= \mathcal{X}^h(J^\infty\pi) \oplus \mathcal{X}^v(J^\infty\pi) \\ \mathcal{X}(\pi_{k,l}) &= \mathcal{X}^h(\pi_{k,l}) \oplus \mathcal{X}^v(\pi_{k,l}) \end{aligned}$$

where the space $\mathcal{X}^v(J^\infty\pi)$ is just $\mathcal{V}(\pi_\infty)$ as defined previously; the spaces $\mathcal{X}^h$ are spaces of total derivatives. The third definition does not give a decomposition of $\mathcal{X}(J^k\pi)$ without the additional structure mentioned in the introduction to this section.

An example:

- If X is a vector field on E then its horizontal and vertical components are the vector fields along $\pi_{1,0}$ given by

  $$\begin{aligned} (X^h)_{j_p^1\phi} &= (j_p^1\phi, X_{\phi(p)})^h = \phi_*\pi_*X_{\phi(p)} \\ (X^v)_{j_p^1\phi} &= X_{\phi(p)} - \phi_*\pi_*X_{\phi(p)}. \end{aligned}$$

  In coordinates, if $X = X^i\frac{\partial}{\partial x^i} + X^\alpha\frac{\partial}{\partial u^\alpha}$ then

  $$(X^h)_{j_p^1\phi} = X^i(\phi(p))\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} + \left.\frac{\partial\phi^\alpha}{\partial x^i}\right|_p \left.\frac{\partial}{\partial u^\alpha}\right|_{\phi(p)}\right)$$

  so that

  $$\begin{aligned} X^h &= X^i\left(\frac{\partial}{\partial x^i} + u^\alpha_{1_i}\frac{\partial}{\partial u^\alpha}\right) = X^i\frac{d}{dx^i} \\ X^v &= (X^\alpha - X^i u^\alpha_{1_i})\frac{\partial}{\partial u^\alpha}. \end{aligned}$$

Note that $(X^v)_{j_p^1\phi}$ is just the vertical tangent vector which I denoted $(\tilde{X}\phi)_p$ in my discussion of the action of X on local sections of π at the end of Section 1.2.

The general coordinate representations for X^h, X^v are

$$\begin{aligned} X^h &= X^i \frac{d}{dx^i} \\ X^v &= \sum_{|I|=0}^{k} (X_I^\alpha - X^i u_{I+1_i}^\alpha) \frac{\partial}{\partial u_I^\alpha} \end{aligned}$$

where $X = X^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} X_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ and, as usual with coordinate representations, I have omitted all the projection maps.

The dual construction to that of the total derivative is that of the contact form; once again this is a pointwise operation. One may describe the duality as follows: at each point $j_p^{k+1}\phi \in J^{k+1}\pi$, the total derivative X^k yields a tangent vector to $J^k\pi$ in the image space of $(j^k\phi)_*$; at each point $j_p^{k+1}\phi$ a contact form σ on $J^{k+1}\pi$ yields the pull-back under $\pi_{k+1,k}^*$ of a cotangent vector to $J^k\pi$ in the kernel of $(j^k\phi)^*$. (I shall refer to this duality again in Proposition 2.3.18.) Here are the definitions.

Definition 2.3.14 *A* contact cotangent vector η *at* $j_p^{k+1}\phi \in J^{k+1}\pi$ *is an element of* $\pi_{k+1,k}^* T_{j_p^k\phi}^* J^k\pi$ *satisfying* $(j^{k+1}\phi)^*\eta = 0$.

This definition does not depend on the particular choice of ϕ. For if ψ is another local section of π satisfying $j_p^{k+1}\psi = j_p^{k+1}\phi$ then, writing η in coordinates as

$$\eta = \eta_i \, dx^i\Big|_{j_p^{k+1}\phi} + \sum_{|I|=0}^{k} \eta_\alpha^I \, du_I^\alpha|_{j_p^{k+1}\phi}$$

one finds that

$$\begin{aligned} (j^{k+1}\psi)^*\eta &= \left(\eta_i + \sum_{|I|=0}^{k} \eta_\alpha^I \left. \frac{\partial^{|I|+1}\psi^\alpha}{\partial x^{I+1_i}} \right|_p \right) dx^i\Big|_p \\ &= \left(\eta_i + \sum_{|I|=0}^{k} \eta_\alpha^I \left. \frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}} \right|_p \right) dx^i\Big|_p \\ &= (j^{k+1}\phi)^*\eta. \end{aligned}$$

It follows immediately from the definition that $(j^k\phi)^*\overline{\eta} = 0$ where $\overline{\eta} \in T_{j_p^k\phi}^* J^k\pi$ satisfies $\pi_{k+1,k}^*\overline{\eta} = \eta$.

Definition 2.3.15 *A differential form* $\sigma \in \bigwedge^1 J^{k+1}\pi$ *is called a* contact form *if, for every* $\phi \in \Gamma_{loc}(\pi)$, $(j^{k+1}\phi)^*(\sigma) = 0$.

Both the above definitions also apply to $J^\infty\pi$ with the convention that $\infty + 1 = \infty$. Note that, when defining a contact *form* on $J^{k+1}\pi$ (k finite) I did not specify explicitly that it had to be horizontal over $J^k\pi$; I shall demonstrate that this must automatically be so in the course of proving the next result.

Proposition 2.3.16 *Suppose* $\sigma \in \bigwedge^1 J^{k+1}\pi$. *Then* σ *is a contact form if, and only if, in local coordinates*

$$\sigma = \sum_{|I|=0}^{k} \sigma_\alpha^I (du_I^\alpha - u_{I+1_i}^\alpha dx^i).$$

Proof For any $\phi \in \Gamma_{loc}(\pi)$, if $0 \leq |I| < k+1$ then

$$\begin{aligned}
(j^{k+1}\phi)^*(du_I^\alpha - u_{I+1_i}^\alpha dx^i) &= d(u_I^\alpha \circ j^{k+1}\phi) - (u_{I+1_i}^\alpha \circ j^{k+1}\phi) d(x^i \circ j^{k+1}\phi) \\
&= d\left(\frac{\partial^{|I|}\phi^\alpha}{\partial x^I}\right) - \frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}} dx^i \\
&= 0
\end{aligned}$$

and so any linear combination (by functions) of these 1-forms is annihilated by $(j^{k+1}\phi)^*$.

To show the reverse implication, I shall first consider the case when k is finite and demonstrate that the contact form σ is an element of $\bigwedge_0^1 \pi_{k+1,k}$. So write σ as

$$\sigma = \sigma_i dx^i + \sum_{|I|=0}^{k} \sigma_\alpha^I du_I^\alpha + \sum_{|J|=k+1} \sigma_\alpha^J du_J^\alpha$$

on some coordinate patch U^{k+1} corresponding to $U \subset M$. Then for any local section ϕ defined on a subset of $\pi(U)$ whose image is contained in U,

$$\begin{aligned}
0 &= (j^{k+1}\phi)^*\sigma \\
&= \left((\sigma_i \circ j^{k+1}\phi) + \sum_{|I|=0}^{k} (\sigma_\alpha^I \circ j^{k+1}\phi) \frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}} \right. \\
&\qquad \left. + \sum_{|J|=k+1} (\sigma_\alpha^J \circ j^{k+1}\phi) \frac{\partial^{|J|+1}\phi^\alpha}{\partial x^{J+1_i}} \right) dx^i
\end{aligned}$$

so that, for each $p \in \text{domain}(\phi)$ and each index i,

$$\sigma_i(j_p^{k+1}\phi) + \sum_{|I|=0}^{k} \sigma_\alpha^I(j_p^{k+1}\phi) \left.\frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}}\right|_p + \sum_{|J|=k+1} \sigma_\alpha^J(j_p^{k+1}\phi) \left.\frac{\partial^{|J|+1}\phi^\alpha}{\partial x^{J+1_i}}\right|_p = 0$$

Now for each p, each pair of indices i and α, and each multi-index J with $|J| = k+1$, choose another section χ defined in a neighbourhood of p satisfying

$$\begin{aligned} j_p^{k+1}\chi &= j_p^{k+1}\phi && \text{(so that each } \tfrac{\partial^{|I|+1}\chi^\alpha}{\partial x^{I+1_i}} = \tfrac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}}\text{)} \\ \frac{\partial^{|K|+1}\chi^\beta}{\partial x^{K+1_i}} &= \frac{\partial^{|K|+1}\phi^\beta}{\partial x^{K+1_i}} && \text{for } \beta \neq \alpha \text{ or } K \neq J \ (|K| = k+1) \\ \frac{\partial^{|J|+1}\chi^\alpha}{\partial x^{J+1_i}} &\neq \frac{\partial^{|J|+1}\phi^\alpha}{\partial x^{J+1_i}}. \end{aligned}$$

(Such a χ may be constructed from ϕ by adding a suitable polynomial function in the coordinate representation.) Subtraction and cancellation then shows that $\sigma_\alpha^J(j_p^{k+1}\phi) = 0$. As $\phi \in \Gamma_{loc}(\pi)$ and $p \in \text{domain}(\phi)$ are arbitrary, $\sigma_\alpha^J = 0$ for $|J| = k+1$, and consequently $\sigma \in \bigwedge_0^1 \pi_{k+1,k}$.

I now have

$$\sigma = \sigma_i dx^i + \sum_{|I|=0}^{k} \sigma_\alpha^I du_I^\alpha$$

and indeed this expression holds for $k = \infty$ as well. Consequently

$$\sigma_i(j_p^{k+1}\phi) + \sum_{|I|=0}^{k} \sigma_\alpha^I(j_p^{k+1}\phi) \left.\frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}}\right|_p = 0$$

which I can rewrite using the coordinate functions on $J^{k+1}\pi$ in the following form:

$$\sigma_i(j_p^{k+1}\phi) + \sum_{|I|=0}^{k} \sigma_\alpha^I(j_p^{k+1}\phi) u_{I+1_i}^\alpha(j_p^{k+1}\phi) = 0$$

giving

$$\sigma_i + \sum_{|I|=0}^{k} \sigma_\alpha^I u_{I+1_i}^\alpha = 0$$

so that

$$\sigma = \sum_{|I|=0}^{k} \sigma_\alpha^I (du_I^\alpha - u_{I+1_i}^\alpha dx^i).$$

∎

Now I have used jet extension maps to define contact forms; conversely, contact forms may be used to characterise which local sections of π_k arise as jet extensions. For if $\psi \in \Gamma_{loc}(\pi_k)$ is going to be j^k of anything, it will have to be $j^k(\pi_{k,0} \circ \psi)$ by Lemma 2.1.18.

Proposition 2.3.17 *If $\psi \in \Gamma_{loc}(\pi_k)$ then $\psi = j^k(\pi_{k,0} \circ \psi)$ if, and only if, for every contact form σ on $J^k\pi$, $\psi^*(\sigma) = 0$.*

Proof If $\psi = j^k(\pi_{k,0} \circ \psi)$ then $\psi^*(\sigma) = 0$ immediately from the definition of a contact form. So suppose conversely that, for every contact form σ, $\psi^*(\sigma) = 0$. I shall show that, for each $p \in \text{domain}(\psi)$,

$$\psi(p) = j^k(\pi_{k,0} \circ \psi)(p)$$

or, in local coordinates,

$$\begin{aligned} u_I^\alpha(\psi(p)) = u_I^\alpha(j_p^k(\pi_{k,0} \circ \psi)) \quad & 0 \le |I| \le k \quad (k \text{ finite}) \\ & 0 \le |I| < \infty \quad (k \text{ infinite}). \end{aligned}$$

Now from $\psi^*(du_I^\alpha - u_{I+1_i}^\alpha dx^i) = 0$, $0 \le |I| < k$, one obtains

$$d(u_I^\alpha \circ \psi) = (u_{I+1_i}^\alpha \circ \psi) dx^i$$

so that

$$\frac{\partial(u_I^\alpha \circ \psi)}{\partial x^i} dx^i = (u_{I+1_i}^\alpha \circ \psi) dx^i$$

and hence

$$\frac{\partial(u_I^\alpha \circ \psi)}{\partial x^i} = u_{I+1_i}^\alpha \circ \psi$$

locally on M. Applying this relationship recursively one finds that, for $0 \le |I| \le k$ (k finite) or $0 \le |I| < \infty$ (k infinite),

$$u_I^\alpha \circ \psi(p) = \left.\frac{\partial^{|I|}\psi^\alpha}{\partial x^I}\right|_p = u_I^\alpha(j_p^k(\pi_{k,0} \circ \psi)).$$

∎

In fact, the above result is quite sharp because the collection of contact forms on $J^k\pi$ is a minimal module of 1-forms (over $C^\infty(J^k\pi)$) which has the required property. To see this, note that locally a basis for the contact forms is given by $du_I^\alpha - u_{I+1_i}^\alpha dx^i$, and that if any single element of this basis is omitted then it will be possible to construct a local section ψ of π_k which pulls the remaining basis contact forms back to zero but is nevertheless not itself a jet extension.

Finally in this section I shall return to the duality between contact forms and total derivatives. Recall first the duality between $\bigwedge_0^1 \pi_{k+1,k}$ (in which the contact forms are a subset) and $\mathcal{X}(\pi_{k+1,k})$ (in which the total derivatives are a subset). I shall use the decomposition $\mathcal{X}(\pi_{k+1,k}) = \mathcal{X}^h(\pi_{k+1,k}) \oplus \mathcal{X}^v(\pi_{k+1,k})$ and will adopt the notation $\bigwedge_C^1(\pi)_{k+1}$ to represent the module of contact forms on $J^{k+1}\pi$.

Proposition 2.3.18 *$\bigwedge_0^1 \pi_{k+1,k} = \bigwedge_0^1 \pi_{k+1} \oplus \bigwedge_C^1(\pi)_{k+1}$; furthermore the modules $\bigwedge_C^1(\pi)_{k+1}$ and $\mathcal{X}^h(\pi_{k+1,k})$ annihilate each other, as do $\bigwedge_0^1 \pi_{k+1}$ and $\mathcal{X}^v(\pi_{k+1,k})$.*

Proof Suppose that $X \in \mathcal{X}^v(\pi_{k+1,k})$, $\sigma \in \bigwedge_0^1 \pi_{k+1}$ and $j_p^{k+1}\phi \in J^{k+1}\pi$. Then $\sigma_{j_p^{k+1}\phi} = \pi_{k+1}^* \eta$ for some $\eta \in T_p^* M$, and so

$$\begin{aligned}(X \lrcorner \sigma)(j_p^{k+1}\phi) &= \pi_k^* \eta(X_{j_p^{k+1}\phi}) \\ &= \eta(\pi_{k*} X_{j_p^{k+1}\phi}) \\ &= 0.\end{aligned}$$

Next, suppose that $X \in \mathcal{X}^h(\pi_{k+1,k})$, $\sigma \in \bigwedge_C^1(\pi)_{k+1}$ and $j_p^{k+1}\phi \in J^{k+1}\pi$. Then $X_{j_p^{k+1}\phi} = (j^k\phi)_* \xi$ for some $\xi \in T_p M$, so that

$$\begin{aligned}(X \lrcorner \sigma)(j_p^{k+1}\phi) &= \sigma_{j_p^{k+1}\phi}((j^k\phi)_* \xi) \\ &= ((j^k\phi)^* \sigma)_p(\xi) \\ &= 0.\end{aligned}$$

Therefore $\bigwedge_0^1 \pi_{k+1} \subset (\mathcal{X}^v(\pi_{k+1,k}))^\circ$ and $\bigwedge_C^1(\pi)_{k+1} \subset (\mathcal{X}^h(\pi_{k+1,k}))^\circ$, and consequently $\bigwedge_0^1 \pi_{k+1,k} = \bigwedge_0^1 \pi_{k+1} \oplus \bigwedge_C^1(\pi)_{k+1}$. ∎

2.4 The Horizontal and Vertical Operators

The considerations in the previous section show that the total derivative/contact form relationship is basic to an understanding of the geometrical structure of a jet bundle, and it is convenient to encapsulate this relationship in two complementary vector-valued forms along the bundle map $\pi_{k+1,k}$. These will be denoted h and v, and when $k = \infty$ they define the natural connection on $J^\infty\pi$; when $k < \infty$ they form the skeleton on which connections are constructed.

Definition 2.4.1 *The horizontal and vertical vector-valued forms along $\pi_{k+1,k}$, denoted $h, v : J^{k+1}\pi \longrightarrow T^*(J^{k+1}\pi) \otimes \pi^* T(J^k\pi)$, are defined by their actions on tangent vectors: if $\xi \in T_{j_p^{k+1}\phi}(J^{k+1}\pi)$ then*

$$\begin{aligned}h_{j_p^{k+1}\phi}(\xi) &= (j_p^{k+1}\phi, \pi_{k+1,k*}\xi)^h \\ v_{j_p^{k+1}\phi}(\xi) &= (j_p^{k+1}\phi, \pi_{k+1,k*}\xi)^v.\end{aligned}$$

Clearly $h + v = \pi_{k+1,k*}$; I shall not normally indicate the particular map along which h or v is defined, because the following lemma is derived immediately from Lemma 2.3.10.

Lemma 2.4.2 *If $j_p^{k+1}\phi \in J^{k+1}\pi$ and $\xi \in T_{j_p^{k+1}\phi}(J^{k+1}\pi)$, and if $0 \leq l < k$, then*

$$\pi_{k,l*}(h_{j_p^{k+1}\phi}(\xi)) = h_{j_p^{l+1}\phi}(\pi_{k+1,l+1*}\xi)$$

and similarly for v.

∎

In coordinates,

$$\begin{aligned} h &= dx^i \otimes \frac{d}{dx^i} \\ v &= \sum_{|I|=0}^{k} (du_I^\alpha - u_{I+1_i}^\alpha dx^i) \otimes \frac{\partial}{\partial u_I^\alpha}. \end{aligned}$$

The derivations of type i_* and d_* associated with h and v will be used on several occasions in later parts of this dissertation. I can summarise their actions on the coordinate functions and coordinate 1-forms as follows:

$$\begin{array}{rclcl} i_h(dx^i) &=& d_h x^i &=& dx^i \\ i_h(du_I^\alpha) &=& d_h u_I^\alpha &=& u_{I+1_i}^\alpha dx^i \\ && d_h dx^i &=& 0 \\ && d_h du_I^\alpha &=& dx^i \wedge du_{I+1_i}^\alpha \end{array}$$

$$\begin{array}{rclcl} i_v(dx^i) &=& d_v x^i &=& 0 \\ i_v(du_I^\alpha) &=& d_v u_I^\alpha &=& du_I^\alpha - u_{I+1_i}^\alpha dx^i \\ && d_v dx^i &=& 0 \\ && d_v du_I^\alpha &=& du_{I+1_i}^\alpha \wedge dx^i. \end{array}$$

I shall call d_h and d_v the horizontal and vertical differentials. A consequence of $h + v = \pi_{k+1,k*}$ is the relationship $d_h + d_v = \pi_{k+1,k}^* \circ d$; this yields the following lemma which shows what happens when the exterior derivative d is taken to the other side of a jet extension map.

Lemma 2.4.3 $d \circ (j^k\phi)^* = (j^{k+1}\phi)^* \circ d_h$.

Proof The exterior derivative d commutes with pull-backs, so

$$\begin{aligned} d \circ (j^k\phi)^* &= (j^k\phi)^* \circ d \\ &= (j^{k+1}\phi)^* \circ \pi_{k+1,k}^* \circ d \\ &= (j^{k+1}\phi)^* \circ (d_h + d_v). \end{aligned}$$

But for any 1-form σ, $v \lrcorner \sigma$ is a contact form, so that $(j^{k+1}\phi)^*(v \lrcorner \sigma) = 0$; therefore $(j^{k+1}\phi)^* \circ i_v = 0$ and so $(j^{k+1}\phi)^* \circ d_v = 0$. ∎

Another important property of the horizontal and vertical differentials is the following.

Lemma 2.4.4 $d_h^2 = d_v^2 = 0$.

Proof The derivations d_h^2, d_v^2 are of type d_* and degree 2 along $\pi_{k+2,k}$. I shall show, using coordinates, that they both vanish on $C^\infty(J^k\pi)$. For d_h^2,

$$d_h^2 f = d_h\left(\frac{df}{dx^i}dx^i\right) = \frac{d^2 f}{dx^j dx^i}dx^j \wedge dx^i = 0$$

so that $d_h^2 = 0$. For d_v^2, I shall first consider $d_v^2 u_I^\alpha$:

$$d_v^2 u_I^\alpha = d_v(du_I^\alpha - u_{I+1_i}^\alpha dx^i) = du_{I+1_i}^\alpha \wedge dx^i - (du_{I+1_i}^\alpha - u_{I+1_i+1_j}^\alpha dx^j)\wedge dx^i = 0$$

and then

$$\begin{aligned} d_v^2 f &= d_v\left(\sum_{|I|=0}^{k}\frac{\partial f}{\partial u_I^\alpha}(du_I^\alpha - u_{I+1_i}^\alpha dx^i)\right) \\ &= \sum_{|I|=0}^{k}\sum_{|J|=0}^{k}\frac{\partial^2 f}{\partial u_J^\beta \partial u_I^\alpha}(du_J^\beta - u_{J+1_j}^\beta dx^j)\wedge(du_I^\alpha - u_{I+1_i}^\alpha dx^i) \\ &= 0. \end{aligned}$$

∎

Corollary 2.4.5 $d_h \circ d_v + d_v \circ d_h = 0$.

Proof $(d_h + d_v)^2 = \pi_{k+2,k}^* \circ d^2$. ∎

From the above, d_h and d_v may be considered as coboundary operators. To show how they fit together, I shall define a canonical splitting of the space $\bigwedge_0^r \pi_{k+1,k}$ which generalises the splitting $\bigwedge_0^1 \pi_{k+1,k} = \bigwedge_0^1 \pi_{k+1} \oplus \bigwedge_C^1(\pi)_{k+1}$. I shall use the following notation:

$\Phi_0^0(\pi_{k+1}) = \bigwedge^0 J^{k+1}\pi$;

$\Phi_0^1(\pi_{k+1}) = \bigwedge_0^1 \pi_{k+1}$; $\Phi_1^0(\pi_{k+1}) = \bigwedge_C^1(\pi)_{k+1}$;

$\theta \in \Phi_s^r(\pi_{k+1}) \subset \bigwedge_0^{r+s}\pi_{k+1,k}$ if, and only if,

1. $\forall X \in \mathcal{X}^h(\pi_{k+1,k})$, $X \lrcorner\, \theta \in \Phi_s^{r-1}(\pi_{k+1})$ (or $X \lrcorner\, \theta = 0$ if $r = 0$);
2. $\forall X \in \mathcal{X}^v(\pi_{k+1,k})$, $X \lrcorner\, \theta \in \Phi_{s-1}^r(\pi_{k+1})$ (or $X \lrcorner\, \theta = 0$ if $s = 0$).

To demonstrate the splitting I shall use the following generalisation of derivations of type i_*:

Definition 2.4.6 *The operator* $h^s : \bigwedge_0^r \pi_{k+1,k} \longrightarrow \bigwedge_0^r \pi_{k+1,k}$ $(0 \le s \le r)$, *is defined by*

$$(X_1, \ldots, X_r) \lrcorner h^s\theta = \sum_{\sigma \in S_{s,r-s}} \varepsilon_\sigma (X^h_{\sigma(1)}, \ldots, X^h_{\sigma(s)}, X_{\sigma(s+1)}, \ldots, X_{\sigma(r)}) \lrcorner \theta$$

where $\theta \in \bigwedge_0^r \pi_{k+1,k}$ *and* $X_1, \ldots, X_r \in \mathcal{X}(\pi_{k+1,k})$.

Proposition 2.4.7 *If* $\theta \in \bigwedge_0^r \pi_{k+1,k}$ *then define* $\theta^{(s)}$ *by* $\theta^{(0)} = h^r\theta$, $\theta^{(s)} = h^{r-s}\theta - h^{r-s+1}\theta$ *for* $1 \le s \le r$. *Then* $\theta^{(s)} \in \Phi^{r-s}_s(\pi_{k+1})$.

Proof By induction on r. The result for $r = 1$ is just Proposition 2.3.18. So suppose it is true for forms in $\bigwedge_0^{r-1} \pi_{k+1,k}$. I shall show that if $\theta \in \bigwedge_0^r \pi_{k+1,k}$ and $X_1 \in \mathcal{X}^h(\pi_{k+1,k})$ that $X_1 \lrcorner \theta^{(s)} = (X_1 \lrcorner \theta)^{(s)}$ for $1 \le s \le r$. If $X_2, \ldots, X_r \in \mathcal{X}(\pi_{k+1,k})$ then

$$\begin{aligned}
& (X_2, \ldots, X_r) \lrcorner (X_1 \lrcorner \theta^{(s)}) \\
= \ & (X_1, \ldots, X_r) \lrcorner h^{r-s}\theta - (X_1, \ldots, X_r) \lrcorner h^{r-s+1}\theta \\
= \ & \sum_{\sigma \in S_{r-s,s}} \varepsilon_\sigma (X^h_{\sigma(1)}, \ldots, X^h_{\sigma(r-s)}, X_{\sigma(r-s+1)}, \ldots, X_{\sigma(r)}) \lrcorner \theta \\
& - \sum_{\sigma \in S_{r-s+1,s-1}} \varepsilon_\sigma (X^h_{\sigma(1)}, \ldots, X^h_{\sigma(r-s+1)}, X_{\sigma(r-s+2)}, \ldots, X_{\sigma(r)}) \lrcorner \theta.
\end{aligned}$$

I shall split each sum into two parts. In the first sum, either $\sigma^{-1}(1) \le r - s$ (in which case $\sigma(1) = 1$, by definition of the "shuffle" group $S_{s,r-s}$) or else $\sigma^{-1}(1) > r - s$ (in which case $\sigma(r - s + 1) = 1$). Similarly, in the second sum either $\sigma(1) = 1$ or $\sigma(r - s + 2) = 1$. But because X_1 is already horizontal,

$$\sum_{\substack{\sigma \in S_{r-s,s} \\ \sigma(r-s+1)=1}} = \sum_{\substack{\sigma \in S_{r-s+1,s-1} \\ \sigma(1)=1}}$$

and so

$$(X_2, \ldots, X_r) \lrcorner (X_1 \lrcorner \theta^{(s)}) = \sum_{\substack{\sigma \in S_{r-s,s} \\ \sigma(1)=1}} - \sum_{\substack{\sigma \in S_{r-s+1,s-1} \\ \sigma(r-s+2)=1}} .$$

But the subgroup of $S_{r-s,s}$ where $\sigma(1) = 1$ is isomorphic to $S_{r-s-1,s}$, and the subgroup of $S_{r-s+1,s-1}$ where $\sigma(r - s + 2) = 1$ is isomorphic to $S_{r-s,s-1}$. So I may write

$$(X_2, \ldots, X_r) \lrcorner (X_1 \lrcorner \theta^{(s)})$$

$$
\begin{aligned}
&= \sum_{\sigma \in S_{r-s-1,s}} \varepsilon_\sigma (X^h_{\sigma(2)}, \ldots, X^h_{\sigma(r-s)}, X_{\sigma(r-s+1)}, \ldots, X_{\sigma(r)}) \lrcorner (X_1 \lrcorner \theta) \\
&\quad \sum_{\sigma \in S_{r-s,s-1}} \varepsilon_\sigma (X^h_{\sigma(2)}, \ldots, X^h_{\sigma(r-s+1)}, X_{\sigma(r-s+2)}, \ldots, X_{\sigma(r)}) \lrcorner (X_1 \lrcorner \theta) \\
&= (X_2, \ldots, X_r) \lrcorner h^{r-s-1}(X_1 \lrcorner \theta) - (X_2, \ldots, X_r) \lrcorner h^{r-s}(X_1 \lrcorner \theta) \\
&= (X_2, \ldots, X_r) \lrcorner (X_1 \lrcorner \theta)^{(s)}
\end{aligned}
$$

so that $X_1 \lrcorner \theta^{(s)} = (X_1 \lrcorner \theta)^{(s)} \in \Phi^{r-s-1}_s(\pi_{k+1,k})$ by the induction hypothesis. The proofs for $s = 0$ and for $X_1 \in \mathcal{X}^v(\pi_{k+1,k})$ are similar. ∎

Corollary 2.4.8 $\bigwedge^r_0 \pi_{k+1,k} = \bigoplus^r_{s=0} \Phi^{r-s}_s(\pi_{k+1,k})$.

∎

As a result of this I can construct (at least part of) the following commutative diagram, for any $k \in \mathbf{N} \cup \{\infty\}$:

$$
\begin{tikzcd}
\Phi^0_0(\pi_k) \arrow[r, "d_v"] \arrow[d, "d_h"'] & \Phi^0_1(\pi_{k+1}) \arrow[d, "-d_h"'] & \cdots & \Phi^0_s(\pi_{k+s}) \arrow[r, "d_v"] \arrow[d, "(-1)^s d_h"'] & \Phi^0_{s+1}(\pi_{k+s+1}) \arrow[d, "(-1)^{s+1} d_h"'] & \cdots \\
\Phi^1_0(\pi_{k+1}) \arrow[r, "d_v"] \arrow[d, "-d_h"'] & \Phi^1_1(\pi_{k+2}) \arrow[d, "d_h"'] & \cdots & \Phi^1_s(\pi_{k+s+1}) \arrow[r, "d_v"] \arrow[d, "(-1)^{s+1} d_h"'] & \Phi^1_{s+1}(\pi_{k+s+2}) \arrow[d, "(-1)^s d_h"'] & \cdots \\
\Phi^2_0(\pi_{k+2}) \arrow[r, "d_v"] & \Phi^2_1(\pi_{k+3}) & \cdots & \Phi^2_s(\pi_{k+s+2}) \arrow[r, "d_v"] & \Phi^2_{s+1}(\pi_{k+s+3}) & \cdots \\
\vdots & \vdots & & \vdots & \vdots & \\
\Phi^{m-1}_0(\pi_{k+m-1}) \arrow[r, "d_v"] \arrow[d, "(-1)^{m-1} d_h"'] & \Phi^{m-1}_1(\pi_{k+m}) \arrow[d, "(-1)^m d_h"'] & \cdots & \Phi^{m-1}_s(\pi_{k+m+s-1}) \arrow[r, "d_v"] \arrow[d, "(-1)^{m+s-1} d_h"'] & \Phi^{m-1}_{s+1}(\pi_{k+m+s}) \arrow[d, "(-1)^{m+s} d_h"'] & \cdots \\
\Phi^m_0(\pi_{k+m}) \arrow[r, "d_v"] & \Phi^m_1(\pi_{k+m+1}) & \cdots & \Phi^m_s(\pi_{k+m+s}) \arrow[r, "d_v"] & \Phi^m_{s+1}(\pi_{k+m+s+1}) & \cdots
\end{tikzcd}
$$

(When $k = \infty$ these are all spaces of forms on $J^\infty\pi$; when $k < \infty$ only that part of the diagram containing $\Phi^r_s(\pi_{k+r+s})$ with $k + r + s \geq 1$ is available.)

Theorem 2.4.9 *The rows and columns of this diagram are locally exact.*

There are several proofs of this result (see [56] for references). Typically, one constructs a suitable homotopy operator to show that d_h is locally exact, and then chases round the diagram to show that d_v is locally exact. I shall not include a proof of this theorem; however in Section 5.3 I shall show how the generalised vertical endomorphisms S_ω which I shall define on an arbitrary jet manifold $J^k\pi$ are precisely the operators given in coordinates by Tulczyjew [67] and used to show local exactness of d_h. ∎

Essentially, the above diagram provides a decomposition of r-forms in $\bigwedge^r_0 \pi_{k+1,k}$ into forms with r dx-factors and $(r-s)$ contact factors. I shall therefore call an element of $\Phi^{r-s}_s(\pi_{k+1})$ an *s-contact r-form on $J^{k+1}\pi$*. However a word of caution is necessary here, for when $r \leq \dim M$ it would seem natural to define a contact r-form on $J^{k+1}\pi$ as an r-form $\theta \in \bigwedge^r J^{k+1}\pi$ satisfying $(j^{k+1}\phi)^*\theta = 0$ for every $\phi \in \Gamma_{loc}(\pi)$. When $r = 1$ this is just a contact form as previously defined, and when $\dim M = 1$ it is indeed a sum of s-contact r-forms for $1 \leq s \leq r$. However, when $\dim M \geq 2$ and $r \geq 2$ there are contact r-forms which cannot be written as sums of elements of $\Phi^{r-s}_s(\pi_{k+1})$ because they are not even elements of $\bigwedge^r_0 \pi_{k+1,k}$.

An example:

- Let π be the trivial bundle $(\mathbf{R}^2 \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates $(x, y; u)$. Then on $J^1\pi$ the 2-form $\theta = d_v du = du_x \wedge dx + du_y \wedge dy$ is a contact form, because

$$\begin{aligned}(j^1\phi)^*\theta &= d\left(\frac{\partial\phi}{\partial x}\right) \wedge dx + d\left(\frac{\partial\phi}{\partial x}\right) \wedge dy \\ &= \frac{\partial^2\phi}{\partial y\,\partial x} dy \wedge dx + \frac{\partial^2\phi}{\partial x\,\partial y} dx \wedge dy \\ &= 0.\end{aligned}$$

I shall therefore adopt a slightly different notation for these "extended" contact r-forms. In particular, I shall wish to use the space of contact r-forms θ that are 1-contact in the sense that $X \lrcorner\, \theta \in \bigwedge^{r-1}_0 \pi_{k+1} = \Phi^{r-1}_0(\pi_{k+1})$ for every $X \in \mathcal{X}^{(v)}(\pi_{k+1,k})$, and I shall denote this space by $\bigwedge^r_C(\pi)_{k+1}$. Then

$$\Phi^{r-1}_1(\pi_{k+1}) = \bigwedge{}^r_C(\pi)_{k+1} \cap \bigwedge{}^r_0 \pi_{k+1,k}.$$

For the final part of this section I shall extend the idea of writing derivations as sums of i_* and d_* derivations, to take account of the additional structure available on a jet bundle.

Definition 2.4.10 *If R, R' are π-related vector-valued r-forms along $\pi_{k+m,k}$ and $\pi_{k+m+1,k+1}$ respectively, define the* derivations along $\pi_{k+m+1,k}$ of type h_* and v_* determined by R *to be*

$$\begin{aligned} h_R &= i_R \circ d_h + (-1)^r d_h \circ i_{R'} \\ v_R &= i_R \circ d_v + (-1)^r d_v \circ i_{R'} \end{aligned}$$

Lemma 2.4.11 $h_R + v_R = d_R \circ \pi^*_{k+1,k}$

∎

Corollary 2.4.12 *Every derivation along $\pi_{k+m+1,k}$ which has been lifted from a derivation along $\pi_{k+m,k}$ may be written as the sum of three derivations, of types i_*, h_* and v_*.*

∎

An example:

- If X is a vector field on $J^{k+1}\pi$ and $f \in C^\infty(J^k\pi)$ then

$$v_X f = i_X d_v f = i_X i_v df = i_{X \lrcorner v} df = d_{X \lrcorner v} f$$

  so the effect of v_X on functions is the same as the effect of d_{X^v}, where $X^v \in \mathcal{X}(\pi_{k+1,k})$ is the vertical component of $\pi_{k+1,k*}X$.

In general, if $f \in C^\infty(J^{k+1}\pi)$ then $v_R f = d_{R \lrcorner v} f$, where one may consider the vector-valued r-form $R \lrcorner v$ to be the "vertical component" of R. By the standard decomposition of derivations I may therefore write $v_R = d_{R \lrcorner v} + i_S$, where S is a vector-valued $(r+1)$-form along $\pi_{k+m+1,k}$. Not surprisingly, S turns out to be the Frölicher-Nijenhuis bracket of v and R.

Lemma 2.4.13 $v_R = d_{R \lrcorner v} + i_{[v,R]}$ *and* $h_R = d_{R \lrcorner h} + i_{[h,R]}$.

Proof From the definitions,

$$\begin{aligned} i_S &= v_R - d_{R \lrcorner v} \\ &= i_R \circ d_v + (-1)^r d_v \circ i_R - i_{R \lrcorner v} \circ d - (-1)^r d \circ i_{R \lrcorner v} \\ &= i_R \circ (i_v \circ d - d \circ i_v) + (-1)^r (i_v \circ d - d \circ i_v) \circ i_R \\ &\quad - i_R \circ i_v \circ d - (-1)^r d \circ i_R \circ i_v \end{aligned}$$

and so

$$\begin{aligned} d_S &= i_S \circ d - (-1)^r d \circ i_S \\ &= -i_R \circ d \circ i_v \circ d + (-1)^r i_v \circ d \circ i_R \circ d - (-1)^r d \circ i_v \circ i_R \circ d \\ &\quad -(-1)^r d \circ i_R \circ i_v \circ d + (-1)^r d \circ i_R \circ d \circ i_v - d \circ i_v \circ d \circ i_R \\ &= (i_v \circ d - d \circ i_v) \circ (i_R \circ d + (-1)^r d \circ i_R) \\ &\quad -(-1)^r (i_R \circ d + (-1)^r d \circ i_R) \circ (i_v \circ d - d \circ i_v) \\ &= d_v \circ d_R - (-1)^r d_R \circ d_v \end{aligned}$$

so that $S = [v, R]$. The result for h_R is similar. ∎

In some ways the vertical differential d_v plays the same rôle on a jet bundle as that normally taken by the exterior derivative d, and so derivatives of type v_* will occasionally appear in later parts of this dissertation. The final result in this section shows how to define the vertical analogue of the Frölicher-Nijenhuis bracket.

Proposition 2.4.14 *Suppose that R^k is a family of π-related vector-valued r-forms along $\pi_{k+m,k}$ and that S^k is a family of π-related vector-valued s-forms along $\pi_{k+l,k}$. Then the operator*

$$v_{R^{k+l+1}} \circ v_{S^k} - (-1)^{rs} v_{S^{k+m+1}} \circ v_{R^k}$$

is a derivation along $\pi_{k+l+m+2,k}$ of type v_*.

Proof Let D be the derivation $(-1)^s i_R \circ v_S - (-1)^{rs} v_S \circ i_R - (-1)^s v_{R \overline{\wedge} S}$, omitting explicit mention of the particular jet map along which each operator is defined. Then

$$D = i_R \circ d_v \circ i_S - (-1)^{rs} i_S \circ d_v \circ i_R - (-1)^{r(s+1)} d_v \circ i_S \circ i_R + (-1)^{r+s} d_v \circ i_R \circ i_S$$

and so is a derivation of type i_*. A straightforward calculation then demonstrates that

$$D \circ d_v + (-1)^{r+s} d_v \circ D = v_R \circ v_S - (-1)^{rs} v_S \circ v_R.$$

∎

As a derivation along $\pi_{k+l+m+1}$ of type i_* and degree $(r+s-1)$, D determines a vector-valued $(r+s)$-form along $\pi_{k+l+m+1,k}$ which I shall denote $[R, S]^v$ and call the *vertical bracket* of R and S. This is the natural generalisation to vector-valued forms of the bracket operation defined for vector fields X, Y by $[X^v, Y^v]$.

Chapter 3

Prolongations

One of the features of jet bundles is that entities defined on the total space E may be prolonged so that new entities are created which act on the corresponding jet manifolds. Another name for this activity would perhaps be differentiation, for it is no more than a formal application of the chain rule to the derivative coordinates. In this chapter I shall show how to prolong bundle maps and vector fields, introducing the important "evolution fields" and showing how they are related to total derivatives and contact forms. As another application I shall describe repeated jets, where a jet bundle π_k is used as a bundle in its own right for the construction of further jet manifolds $J^l\pi_k$, and show how certain maps associated naturally with repeated jets appear as prolongations. All the material in this chapter is standard.

3.1 Prolongations of Bundle Maps

Suppose that (E, π, M) and (F, ρ, N) are bundles, and that $(f, \overline{f})$ is a bundle morphism from π to ρ where, in particular, $\overline{f}$ a diffeomorphism.

Definition 3.1.1 *The k-th prolongation of f is the map $f^k : J^k\pi \longrightarrow J^k\rho$ defined by*

$$f^k(j^k_p\phi) = j^k_{\overline{f}(p)}\tilde{f}(\phi)$$

where $\tilde{f}(\phi) \in \Gamma_{loc}(\rho)$ is specified by $\tilde{f}(\phi)(p) = f(\phi(\overline{f}^{-1}(p)))$ whenever $p \in \overline{f}(\text{domain}(\phi))$.

This definition does not depend on the particular choice of ϕ, for $j^k_{\overline{f}(p)}\tilde{f}(\phi)$ only involves the values of ϕ and its derivatives of order k or less at p. I can also rewrite the definition using k-jet extension maps: for each $p \in \text{domain}(\phi)$,

$$(f^k \circ j^k\phi)(p) = f^k(j^k_p\phi) = j^k_{\overline{f}(p)}\tilde{f}(\phi) = (j^k\tilde{f}(\phi) \circ \overline{f})(p)$$

so that

$$\begin{aligned} f^k \circ j^k\phi &= j^k\tilde{f}(\phi) \circ \overline{f} \\ &= j^k(f \circ \phi \circ \overline{f}^{-1}) \circ \overline{f} \end{aligned}$$

giving, rather suggestively,

$$f^k \circ j^k\phi \circ \overline{f}^{-1} = j^k(f \circ \phi \circ \overline{f}^{-1}).$$

The fact that f^k is C^∞ follows from applying the chain rule and observing that the derivative coordinates of $f^k(j^k_p\phi)$ are polynomials in the derivatives of f and the derivative coordinates of $j^k_p\phi$.

Some examples:

- Suppose π,ρ are the trivial bundles $(\mathbf{R} \times F, pr_1, \mathbf{R})$ and $(\mathbf{R} \times G, pr_1, \mathbf{R})$ respectively, that $g : F \longrightarrow G$ is a map, and that $f = id_{\mathbf{R}} \times g : \mathbf{R} \times F \longrightarrow \mathbf{R} \times G$ (so that $(f, id_{\mathbf{R}})$ is a bundle morphism). Then remembering the identifications $J^1\pi \cong \mathbf{R} \times TF$ and $J^1\rho \cong \mathbf{R} \times TG$, one finds that $f^1 = id_{\mathbf{R}} \times g_*$: for $f^1(j^1_t\phi) = j^1_t(f \circ \phi) = (t, g_*[\phi]_t)$ where $[\phi]_t$ is the tangent vector at $pr_2 \circ \phi(t) \in F$ determined by the curve $pr_2 \circ \phi$.

- Suppose now that π,ρ are the trivial bundles $(M \times \mathbf{R}, pr_1, M)$ and $(N \times \mathbf{R}, pr_1, N)$ respectively, and that $f : M \longrightarrow N$ is a diffeomorphism so that $(f \times id_{\mathbf{R}}, f)$ is a bundle isomorphism. Then remembering the identification $J^1\pi \cong T^*M \times \mathbf{R}$, $J^1\rho \cong T^*N \times \mathbf{R}$, one finds that $(f \times id_{\mathbf{R}})^1 = (f^{-1})^* \times id_{\mathbf{R}}$: for $j^1_p\phi = (d\overline{\phi}_p, \overline{\phi}(p))$, whereas

$$\begin{aligned} (f \times id_{\mathbf{R}})^1(j^1_p\phi) &= j^1_{f(p)}((f \times id_{\mathbf{R}}) \circ \phi \circ f^{-1}) \\ &= j^1_{f(p)}(id_N, \overline{\phi} \circ f^{-1}) \\ &= (d(\overline{\phi} \circ f^{-1})_{f(p)}, \overline{\phi}(p)) \\ &= (f^{-1*} d\overline{\phi}_p, \overline{\phi}(p)) \end{aligned}$$

  using as I have previously the notation $\overline{\phi} = pr_2 \circ \phi$ for this particular type of example.

Prolongations of bundle maps have certain desirable properties.

Lemma 3.1.2 $(g \circ f)^k = g^k \circ f^k$ *and* $(id_E)^k = id_{J^k\pi}$.

Proof Directly from the definitions, using the relationships $\overline{g \circ f} = \overline{g} \circ \overline{f}$ and $\widetilde{g \circ f} = \widetilde{g} \circ \widetilde{f}$. For every $j_p^k \phi \in J^k\pi$,

$$\begin{aligned}
(g \circ f)^k(j_p^k \phi) &= j^k_{\overline{g \circ f}(p)}(\widetilde{g \circ f})(\phi) \\
&= j^k_{\overline{g}(\overline{f}(p))}\widetilde{g}(\widetilde{f}(\phi)) \\
&= g^k(j^k_{\overline{f}(p)}\widetilde{f}(\phi)) \\
&= g^k(f^k(j_p^k \phi).
\end{aligned}$$

∎

Lemma 3.1.3 *$(f^k, \overline{f})$ is a bundle morphism $(J^k\pi, \pi_k, M) \longrightarrow (J^k\rho, \rho_k, N)$.*

Proof Directly from the definition. For every $j_p^k \phi \in J^k\pi$,

$$\begin{aligned}
\rho_k \circ f^k(j_p^k \phi) &= \rho_k(j^k_{\overline{f}(p)}\widetilde{f}(\phi)) \\
&= \overline{f}(p) \\
&= \overline{f} \circ \pi_k(j_p^k \phi)
\end{aligned}$$

∎

.

I can now characterise f^k by the even more suggestive description

$$\widetilde{f^k}(j^k \phi) = j^k \widetilde{f}(\phi).$$

In fact an alternative notation would be to write $J^k f$ instead of f^k, so that J^k might be considered as a functor. However, one must be careful about this: J^k is defined on the category of bundles and *bundle maps over diffeomorphisms*, rather than the usual category of bundles and general bundle maps.

I can also consider how prolongations relate to the bundle $\pi_{k,0}$ rather than π_k.

Lemma 3.1.4 *(f^k, f) is a bundle morphism*

$$(J^k\pi, \pi_{k,0}, E) \longrightarrow (J^k\rho, \rho_{k,0}, F).$$

Proof Again directly from the definitions. For every $j_p^k \phi \in J^k\pi$,

$$\begin{aligned}
\rho_{k,0} \circ f^k(j_p^k \phi) &= \rho_{k,0}(j^k_{\overline{f}(p)}\widetilde{f}(\phi)) \\
&= \widetilde{f}(\phi)(\overline{f}(p)) \\
&= f \circ \phi \circ \overline{f}^{-1} \circ \overline{f}(p) \\
&= f \circ \phi(p) \\
&= f \circ \pi_{k,0}(j_p^k \phi)
\end{aligned}$$

∎

.

In other words, this whole diagram commutes:

$$\begin{tikzcd}
J^k\pi \arrow[r, "f^k"] \arrow[d, "\pi_{k,0}"'] & J^k\rho \arrow[d, "\rho_{k,0}"] \\
E \arrow[r, "f"] \arrow[d, "\pi"'] & F \arrow[d, "\rho"] \\
M \arrow[r, "\overline{f}"] & N
\end{tikzcd}$$

It is clear from the definition that if f is a bundle isomorphism from π to itself then f^k preserves the graphs of jet extensions in $J^k\pi$, for the image of $j^k\phi$ is mapped to the image of $j^k\tilde{f}(\phi)$. One might therefore ask whether prolongations of bundle isomorphisms of π were the only diffeomorphisms of $J^k\pi$ with this property. In fact, when the fibre dimension of π is greater than one then they are the only diffeomorphisms defined on *all* of $J^k\pi$ which have this property. However there exist diffeomorphisms defined on open subsets of $J^k\pi$ which preserve the graphs of jet extensions and which project onto E but not onto M (in other words, they do not preserve the distinction between dependent and independent variables). For a discussion of this, see [44]. There is also a theory of extended jets, where the fibration π is ignored and one considers jets of arbitrary local embeddings of M in E: see [55]. I shall not go into these matters here.

3.2 Prolongations of Vector Fields

In this section I shall use the technique of taking vertical representatives to construct the prolongations of vector fields on E and vector fields along $\pi_{k,0}$. The reason for adopting this technique is that the prolongation of a vertical vector field is easy to describe intrinsically, whereas a simple description for non-vertical fields only seems available for projectable vector fields on E. I shall show later that these definitions are equivalent. First, however, I shall prove a result about certain bundles of tangent vectors.

From a bundle (E, π, M) I can construct two bundles involving TE: (TE, τ_E, E) and $(TE, \pi \circ \tau_E, M)$. However, neither $J^k\tau_E$ nor $J^k(\pi \circ \tau_E)$ is equivalent to $T(J^k\pi)$: in particular, the dimensions don't match. For instance, if M is one-dimensional and E has one-dimensional fibres then TE is four-dimensional, so that $\dim J^1\tau_E = 8$, $\dim J^1(\pi \circ \tau_E) = 7$, whereas $\dim T(J^1\pi) = 6$. However, there is a relation between jets and vertical vectors which may be expressed colloquially as $J^kV = VJ^k$.

Proposition 3.2.1 *Given the two bundles $(V\pi, \nu_\pi, M)$ and $(J^k\pi, \pi_k, M)$ then $J^k\nu_\pi \cong V\pi_k$ for $1 \leq k \leq \infty$ (where the diffeomorphism projects onto the identity on M).*

Proof Consider the maps $\gamma : U \times \mathbf{R} \longrightarrow E$ satisfying $\pi \circ \gamma(p,t) = p$, where $U \subset M$ is any open submanifold (so that γ represents a one-parameter family of local sections of π). For each such map γ, define $\gamma_t \in \Gamma_{loc}(\pi)$ by $\gamma_t(p) = \gamma(p,t)$ $(t \in \mathbf{R})$ and $\gamma_p : \mathbf{R} \longrightarrow E$ by $\gamma_p(t) = \gamma(p,t)$ $p \in U$.

Let $t \in \mathbf{R}$; then $j^k_p\gamma_t \in J^k\pi$, so the map $t \longmapsto j^k_p\gamma_t$ is a curve in $J^k\pi$ which projects onto a single point $p \in M$, and so defines a vertical tangent vector to $J^k\pi$ at $j^k_p\gamma_0$ which will be an element of $V\pi_k$. Different maps γ define the same element of $V\pi_k$ if, and only if, the value and "space" derivatives of γ, and the first "time" derivative of these quantities (when evaluated at $(p,0) \in M \times \mathbf{R}$) are equal.

Now let $p \in U$; then γ_p is a curve in E projecting to a single point $p \in M$, and so defines a vertical tangent vector to E at $\gamma_p(0)$ which I shall denote $[\gamma_p]$. The map $p \longmapsto [\gamma_p]$ is a local section of ν_π, so for each point $p \in U$ there is a corresponding element $j^k_p(q \longmapsto [\gamma_q])$ of $J^k\nu_\pi$. Different maps γ define the same element of $J^k\nu_\pi$ if, and only if, the value and first "time" derivative of γ, and the "space" derivatives of these quantities (when evaluated at $(p,0) \in M \times \mathbf{R}$) are equal.

Since it is immaterial whether the space derivatives are taken before or after the time derivative, these two equivalence relations on maps $U \times \mathbf{R} \longrightarrow E$ $(U \subset M)$ are the same. Since every element of $J^k\nu_\pi$ and every element of $V\pi_k$ may be defined by one of these maps, the correspondence between $J^k\nu_\pi$ and $V\pi_k$ is bijective. Writing it in coordinates shows that it is a diffeomorphism projecting onto the identity. ∎

An example:

- Let π be the bundle $(\mathbf{R}^2 \times \mathbf{R}, pr_1, \mathbf{R}^2)$. Coordinates on $V\pi$ are $(x, y; u, \dot{u})$, so coordinates on $J^2\nu_\pi$ are

$$(x, y; u, \dot{u}; u_x, u_y, \dot{u}_x, \dot{u}_y; u_{xx}, u_{xy}, u_{yy}, \dot{u}_{xx}, \dot{u}_{xy}, \dot{u}_{yy}).$$

On the other hand, coordinates on $J^2\pi$ are $(x, y; u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ so coordinates on $V\pi_2$ are

$$(x, y; u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}; \dot{u}, \dot{u}_x, \dot{u}_y, \dot{u}_{xx}, \dot{u}_{xy}, \dot{u}_{yy})$$

which (apart from the order of presentation) are clearly the same.

Definition 3.2.2 *A* vertical generalised vector field *is a section X of the bundle $\pi^*_{k,0}V\pi \longrightarrow J^k\pi$.*

In coordinates, a vertical generalised vector field may be written as

$$X = X^\alpha \frac{\partial}{\partial u^\alpha} \qquad X^\alpha \in C^\infty(J^k\pi)$$

or, more explicitly,

$$X_{j^k_p\phi} = X^\alpha(j^k_p\phi) \left.\frac{\partial}{\partial u^\alpha}\right|_{\phi(p)} .$$

When $k = 0$ this is of course just a vertical vector field on E. For a further discussion of vertical generalised vector fields, see [42].

When k is finite, any vertical generalised vector field $X : J^k\pi \longrightarrow V\pi$ gives rise to other vertical generalised vector fields defined higher up, namely $X \circ \pi_{l,k} : J^l\pi \longrightarrow V\pi$. Equally, X itself may have arisen by the same process from a vertical generalised vector field $J^m\pi \longrightarrow V\pi$, where $0 \leq m < k$ (here, the case $k = \infty$ is included). I can use this to define the order of X.

Definition 3.2.3 *If $X : J^k\pi \longrightarrow V\pi$ is a vertical generalised vector field then the* order *of X is the smallest $m \in \mathbf{N}$ such that there is a vertical generalised vector field $Y : J^m\pi \longrightarrow V\pi$ satisfying $X = Y \circ \pi_{k,m}$. If k is infinite and there is no such m then X is said to have* unbounded order.

$$\begin{tikzcd}
J^k\pi \arrow[dr, "X"] \arrow[dd, "\pi_{k,m}"'] & \\
& TE \\
J^m\pi \arrow[ur, "Y"'] &
\end{tikzcd}$$

Note that the above definition is global. There may be a neighbourhood of a point $j_p^k\phi$ in which X may be lifted from a lower-order jet manifold than $J^m\pi$.

I shall now show how to prolong vertical generalised vector fields.

Definition 3.2.4 *If $X : J^k\pi \longrightarrow V\pi$ is a vertical generalised vector field on E then the l^{th}* prolongation of X *is the map*

$$X^l : J^{k+l}\pi \longrightarrow J^l\nu_\pi$$

defined by

$$X^l_{j_p^{k+l}\phi} = j_p^l(X \circ j^k\phi).$$

This definition is just a special case of the prolongation of the bundle map (X^l, id_M) from $\pi_{k,l}$ to $(\nu_\pi)_l$.

Lemma 3.2.5 *With the identification $J^l\nu \cong V\pi_l$, X^l becomes a vector field along $\pi_{k+l,l}$.*

Proof I have to show that $\tau_{J^l\pi} \circ X^l = \pi_{k,l}$. So let $\phi \in \Gamma_{loc}(\pi)$, $p \in \mathrm{domain}(\phi)$. Then $X \circ j^k\phi(q)$ $(q \in \mathrm{domain}(\phi))$ is a smoothly varying family of tangent vectors at points of a submanifold of E passing through $\phi(p)$, and so may be extended to a vertical vector field in a neighbourhood of $\phi(p)$. Let $\psi : U \times I \longrightarrow E$, U an open submanifold of E, I an open interval, $(\phi(p), 0) \in U \times I$, be the flow of this vector field, and let $\gamma : \pi(U) \times I \longrightarrow E$ be defined by $\gamma(q,t) = \psi(\phi(q), t)$. Then $[t \longmapsto \gamma(q,t)] = X \circ j^k\phi(q)$ and $\gamma(q, 0) = \phi(q)$. Using the diffeomorphism of Proposition 3.2.1,

$$\begin{aligned} j_p^l(X \circ j^k\phi) &= j_p^l(q \longmapsto [t \longmapsto \gamma(q,t)]) \\ &= [t \longmapsto j_p^l(q \longmapsto \gamma(q,t))] \end{aligned}$$

and this tangent vector to $J^l\pi$ is based at $j_p^l(q \longmapsto \gamma(q,0)) = j_p^l\phi$. ∎

With my usual understanding about projection maps, this result also shows that the infinite prolongation of a vertical generalised vector field is actually an ordinary vertical vector field on the manifold $J^\infty\pi$. I shall call such a vector field an *evolution* field. The reason for this terminology will become apparent when I consider the action of such a vector field on sections.

As one might expect, prolongations behave well with respect to the jet projections.

Lemma 3.2.6 *If $0 \leq m < l \leq \infty$ then $\pi_{l,m*}X^l_{j_p^{k+l}\phi} = X^m_{j_p^{k+m}\phi}$.*

Proof Let $\gamma : \pi(U) \times I \longrightarrow E$ be the function defined in Lemma 3.2.5. Then

$$\begin{aligned}
\pi_{l,m*}X^l_{j_p^{k+l}\phi} &= \pi_{l,m*}(j_p^l(X \circ j^k\phi)) \\
&= \pi_{l,m*}(j_p^l(q \longmapsto [t \longmapsto \gamma(q,t)])) \\
&= \pi_{l,m*}[t \longmapsto j_p^l(q \longmapsto \gamma(q,t))] \\
&= [t \longmapsto \pi_{l,m}(j_p^l(q \longmapsto \gamma(q,t)))] \\
&= [t \longmapsto j_p^m(q \longmapsto \gamma(q,t))] \\
&= j_p^m(q \longmapsto [t \longmapsto \gamma(q,t)]) \\
&= j_p^m(X \circ j^k\phi) \\
&= X^m_{j_p^{k+m}\phi}.
\end{aligned}$$

∎

Corollary 3.2.7 *If* $0 \leq m < l \leq \infty$ *then*

$$\pi^*_{k+l,k+m} \circ d_{X^m} = d_{X^l} \circ \pi^*_{l,m}.$$

Proof If $f \in C^\infty(J^m\pi)$, $j_p^{k+l}\phi \in J^{k+l}\pi$ then

$$\begin{aligned}
(\pi^*_{k+l,k+m} \circ d_{X^m})(f)(j_p^{k+l}\phi) &= d_{X^m}f(j_p^{k+m}\phi) \\
&= X^m_{j_p^{k+m}\phi}(f) \\
&= (\pi_{l,m*}X^l_{j_p^{k+l}\phi})f \\
&= X^l_{j_p^{k+l}\phi}(\pi^*_{l,m}f) \\
&= (d_{X^l} \circ \pi^*_{l,m})(f)(j_p^{k+l}\phi).
\end{aligned}$$

∎

Proposition 3.2.8 *Let* $X : J^k\pi \longrightarrow V\pi$, $Y : J^m\pi \longrightarrow V\pi$ *be vertical generalised vector fields. Then the Frölicher-Nijenhuis bracket* $[X^m, Y^k] : J^{k+m}\pi \longrightarrow V\pi$ *is a vertical generalised vector field.*

Proof X^m, X are π-related in the sense of Definition 1.4.7, as are Y^k, Y so that $[X^m, Y^k]$ is defined as a vector field along $\pi_{k+m,0}$. If $f \in \pi^*C^\infty(M)$ then $d_X f = d_Y f = 0$ so that $d_{[X^m,Y^k]}f = 0$ and hence $[X^m, Y^k]$ is vertical over M and therefore defines a generalised vector field. ∎

I shall now establish a coordinate formula for the prolongation of a vertical generalised vector field $X : J^k\pi \longrightarrow V\pi$, where locally $X = X^\alpha \frac{\partial}{\partial u^\alpha}$. Once again I shall use the map γ defined in Lemma 3.2.5, and I shall also define the functions γ^α, γ^α_t by $\gamma^\alpha = u^\alpha \circ \gamma$ and $\gamma^\alpha_t(q) = u^\alpha(\gamma(q,t))$. So suppose

$X^m = \sum_{0\leq|I|\leq m} X_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ where $X_I^\alpha \in C^\infty(J^{k+m}\pi)$. One obtains the long chain of equalities

$$
\begin{aligned}
X_I^\alpha(j_p^{k+m}\phi) &= (d_{X^m}(u_I^\alpha))(j_p^{k+m}\phi) \\
&= X^m_{j_p^{k+m}\phi}(u_I^\alpha) \\
&= (j_p^m(q \longmapsto X_{j_q^k\phi}))(u_I^\alpha) \\
&= (j_p^m(q \longmapsto [t \longmapsto \gamma(q,t)]))(u_I^\alpha) \\
&= [t \longmapsto j_p^m(q \longmapsto \gamma(q,t))](u_I^\alpha) \\
&= \left.\frac{d}{dt}\right|_{t=0} (t \longmapsto u_I^\alpha(j_p^m(q \longmapsto \gamma(q,t)))) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(t \longmapsto \left.\frac{\partial^{|I|}\gamma_t^\alpha}{\partial x^I}\right|_p\right) \\
&= \left.\frac{\partial^{|I|+1}\gamma^\alpha}{\partial t\,\partial x^I}\right|_{t=0;p} \\
&= \left.\frac{\partial^{|I|}}{\partial x^I}\right|_p \left.\frac{\partial\gamma^\alpha}{\partial t}\right|_{t=0} \\
&= \left.\frac{\partial^{|I|}}{\partial x^I}\right|_p (q \longmapsto X^\alpha(j_q^k\phi)) \\
&= \left.\frac{\partial^{|I|}}{\partial x^I}\right|_p (X^\alpha \circ j^k\phi) \\
&= \left(\frac{\partial^{|I|}}{\partial x^I} \circ (j^k\phi)\right)(X^\alpha)(p) \\
&= \left((j^{k+|I|}\phi)^* \circ \frac{d^{|I|}}{dx^I}\right)(X^\alpha(p)) \\
&= \frac{d^{|I|}X^\alpha}{dx^I}(j_p^{k+|I|}\phi) \\
&= \left(\frac{d^{|I|}X^\alpha}{dx^I} \circ \pi_{k+m,k+|I|}\right)(j_p^{k+m}\phi)
\end{aligned}
$$

so that $X_I^\alpha = \pi^*_{k+m,k+|I|}\left(\frac{d^{|I|}X^\alpha}{dx^I}\right)$. Of course, I shall always ignore the pullback maps and write

$$X^m = \sum_{|I|=0}^{m} \frac{d^{|I|}X^\alpha}{dx^I}\frac{\partial}{\partial u_I^\alpha}.$$

The same argument holds when m is infinite.

I can use this coordinate representation to prove the important result that vertical generalised vector fields commute with holonomic lifts.

Proposition 3.2.9 *If $X : J^k\pi \longrightarrow V\pi$ is a vertical generalised vector field on E, and Y is a vector field on M, then for any $m \in \mathbf{N}$*

$$[Y^{k+m}, X^{m+1}] = 0$$

Proof In local coordinates. Suppose that $X = X^\alpha \frac{\partial}{\partial u^\alpha}$ and $Y = Y^i \frac{\partial}{\partial x^i}$. Then for $f \in C^\infty(J^m\pi)$,

$$d_{Y^{k+m}} d_{X^m} f = Y^i \frac{d}{dx^i} \left(\sum_{|I|=0}^{m} \frac{d^{|I|} X^\alpha}{dx^I} \frac{\partial f}{\partial u_I^\alpha} \right)$$

so that

$$\begin{aligned} d_{Y^{k+m}} d_{X^m} x^j &= 0, \\ d_{Y^{k+m}} d_{X^m} u_J^\beta &= Y^i \frac{d^{|J|+1} X^\beta}{dx^{J+1_i}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d_{X^{m+1}} d_{Y^m} f &= \sum_{|I|=0}^{m+1} \frac{d^{|I|} X^\alpha}{dx^I} \frac{\partial}{\partial u_I^\alpha} \left(Y^i \frac{df}{dx^i} \right) \\ &= Y^i \sum_{|I|=0}^{m+1} \frac{d^{|I|} X^\alpha}{dx^I} \frac{\partial}{\partial u_I^\alpha} \left(\frac{df}{dx^i} \right) \end{aligned}$$

because the functions Y^i have been pulled back from M, so that

$$\begin{aligned} d_{X^{m+1}} d_{Y^m} x^j &= 0, \\ d_{X^{m+1}} d_{Y^m} u_J^\beta &= Y^i \frac{d^{|J|+1} X^\beta}{dx^{J+1_i}}. \end{aligned}$$

Consequently $d_{Y^{k+m}} \circ d_{X^m} = d_{X^{m+1}} \circ d_{Y^m}$. ∎

A similar argument applies when $m = \infty$; the result is then

$$[Y^\infty, X^\infty] = 0$$

so that, on $J^\infty\pi$, every holonomic lift commutes with every evolution field. It easily follows that, on $J^\infty\pi$, if Y is a total derivative and X is an evolution field then $[Y, X]$ is again a total derivative, so that the Lie derivative action of an evolution field preserves the module of total derivatives.

The converse of the last proposition gives a characterisation of those vertical vector fields along $\pi_{k+m,m}$ which are m-prolongations of vertical generalised vector fields.

Proposition 3.2.10 *Suppose the vector fields $X'' \in \mathcal{X}^v(\pi_{k+m+1,m+1})$ and $X' \in \mathcal{X}^v(\pi_{k+m,m})$ are π-related. If, for every vector field $Y \in \mathcal{X}(M)$,*

$$[Y^{k+m}, X''] = 0$$

then there is a vertical generalised vector field $X \in \mathcal{X}(\pi_{k,0})$ such that $X' = X^m$, $X'' = X^{m+1}$.

Proof In local coordinates. Take for Y the successive coordinate vector fields $\frac{\partial}{\partial x^i}$. Then if $X' = \sum_{|I|=0}^{m} X_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ and $X'' = \sum_{|I|=0}^{m+1} X_I^\alpha \frac{\partial}{\partial u_I^\alpha}$, where $X_I^\alpha \in C^\infty(J^{k+m}\pi)$ for $0 \leq |I| \leq m$ and $X_I^\alpha \in C^\infty(J^{k+m+1}\pi)$ for $|I| = m+1$,

$$d_{Y^{k+m}} \circ d_{X'} - d_{X''} \circ d_{Y^m} = 0$$

so that

$$\begin{aligned}
\sum_{|I|=0}^{m} \frac{dX_I^\alpha}{dx^i} \frac{\partial}{\partial u_I^\alpha} &= \sum_{|I|=0}^{m+1} X_I^\alpha \sum_{|J|=0}^{m} \frac{\partial u_{J+1_i}^\beta}{\partial u_I^\alpha} \frac{\partial}{\partial u_J^\beta} \\
&= \sum_{|J|=0}^{m} X_{J+1_i}^\alpha \frac{\partial}{\partial u_J^\alpha}
\end{aligned}$$

and hence, equating coefficients,

$$X_{I+1_i}^\alpha = \frac{dX_I^\alpha}{dx^i} \qquad 0 \leq |I| \leq m$$

so that

$$X_I^\alpha = \frac{d^{|I|} X^\alpha}{dx^I} \qquad 0 \leq |I| \leq m+1$$

and therefore $X \in \mathcal{X}(\pi_{k,0})$ defined by $X_{j_p^k \phi} = \pi_{m,0*} X'_{j_p^{k+m}\phi}$ may be prolonged to give X' and X''. ∎

Again there is a corresponding result for infinite prolongations: if $X \in \mathcal{V}(\pi_\infty)$ satisfies $[Y^\infty, X]$ for every $Y \in \mathcal{X}(M)$ (where the bracket may now be regarded as the ordinary Lie bracket of vector fields on $J^\infty\pi$) then X is an evolution field.

The dual result to the two preceding propositions is that vertical generalised vector fields preserve the module of contact forms.

Proposition 3.2.11 *If $X' \in \mathcal{X}^v(\pi_{k+m,m})$ is the prolongation of a vertical generalised vector field and $\sigma \in \bigwedge^1 J^m\pi$ is a contact form then $d_{X'}\sigma \in \bigwedge^1 J^{k+m}\pi$ is a contact form. Conversely, if $X' \in \mathcal{X}^v(\pi_{k+m,m})$ has the property that $d_{X'}\sigma$ is a contact form on $J^{k+m}\pi$ whenever σ is a contact form on $J^m\pi$ then X' is the prolongation of a vertical generalised vector field.*

Proof Suppose first that $X' = X^m$ where X is a vertical generalised vector field, and that σ is a contact form. Then $\sigma \in \bigwedge_0^1 \pi_{m,m-1}$, so given a point $j_p^m\phi \in J^m\pi$ there is a cotangent vector $\eta \in T^*(J^{m-1}\pi)$ such that $\sigma_{j_p^m\phi} = \pi_{m,m-1}^*\eta$. Then

$$\begin{aligned}
(i_{X^m}\sigma)_{j_p^{k+m}\phi} &= \sigma_{j_p^m\phi}(X^m_{j_p^{k+m}\phi}) \\
&= \pi^*_{m,m-1}\eta(X^m_{j_p^{k+m}\phi}) \\
&= \eta(\pi_{m,m-1*}X^m_{j_p^{k+m}\phi}) \\
&= \eta(X^{m-1}_{j_p^{k+m-1}\phi})
\end{aligned}$$

so that $i_{X^m}\sigma \in \pi^*_{k+m,k+m-1}C^\infty(J^{k+m-1}\pi)$, and consequently $di_{X^m}\sigma \in \bigwedge_0^1 \pi_{k+m,k+m-1}$. By a similar argument $i_{X^m}d\sigma \in \bigwedge_0^1 \pi_{k+m,k+m-1}$ and therefore $d_{X^m}\sigma \in \bigwedge_0^1 \pi_{k+m,k+m-1}$. If now Y is an arbitrary vector field on M then

$$d_{X^m}(Y^{m-1} \lrcorner\, \sigma) = [X^m, Y^{k+m-1}] \lrcorner\, \sigma + Y^{k+m-1} \lrcorner\, d_{X^m}\sigma.$$

Now $Y^{m-1} \lrcorner\, \sigma = 0$ and $[X^m, Y^{k+m-1}] = 0$, so that $Y^{k+m-1} \lrcorner\, d_{X^m}\sigma = 0$, and so $d_{X^m}\sigma$ is a contact form.

Conversely, suppose $X' \in \mathcal{X}^v(\pi_{k+m,m})$ satisfies the property given in the proposition. Then because $d_{X'}\sigma$ is a contact form on $J^{k+m}\pi$ and so is an element of $\bigwedge_0^1 \pi_{k+m,k+m-1}$ for each contact form σ on $J^k\pi$, an argument in coordinates shows that X' must be π-related to a vector field along $\pi_{k+m-1,m-1}$. So if Y is an arbitrary vector field on M then

$$d_{X'}(Y^{m-1} \lrcorner\, \sigma) = [X', Y^{k+m-1}] \lrcorner\, \sigma + Y^{k+m-1} \lrcorner\, d_{X'}\sigma$$

and both $Y^{m-1} \lrcorner\, \sigma$ and $Y^{k+m-1} \lrcorner\, d_{X'}\sigma$ are zero, so that $[X', Y^{k+m-1}] \lrcorner\, \sigma$ is zero: since σ is arbitrary, $[X', Y^{k+m-1}] \in \mathcal{X}^h(\pi_{k+m,m-1})$ and since X' is vertical and Y^{k+m-1} is projectable, $[X', Y^{k+m-1}]$ is also vertical and hence is zero. Since Y is arbitrary, X is therefore the prolongation of a vertical generalised vector field. ∎

Again this result has a simpler formulation on $J^\infty\pi$: it is that the vertical (over M) vector fields which preserve the module of contact forms under the Lie derivative are precisely the evolution fields.

I shall now use the operation of taking vertical representatives to define the prolongation of an arbitrary, not necessarily vertical, vector field along $\pi_{k,0}$. If $k > 0$, the result will be a vector field along $\pi_{k+l,l}$. Prolonging a vector field on E would appear to give a new vector field along $\pi_{l+1,l}$; however, I shall show that one may actually construct a vector field on $J^l\pi$ by this process.

Definition 3.2.12 *If $X \in \mathcal{X}(\pi_{k,0})$, $k > 0$, the* l-th prolongation of X *is the vector field $X^l \in \mathcal{X}(\pi_{k+l,l})$ defined by*

$$(X^l)_{j_p^{k+l}\phi} = (X^v)^l_{j_p^{k+l}\phi} + (j^l\phi)_*\pi_* X_{j_p^k\phi}.$$

Definition 3.2.13 *If $X \in \mathcal{X}(E)$, the* l-th prolongation of X *is the vector field $X \in \mathcal{X}(J^l\pi)$ defined by*

$$(X^l)_{j_p^l\phi} = (X^v)^l_{j_p^{l+1}\phi} + (j^l\phi)_*\pi_* X_{\phi(p)}.$$

In the second definition, I must show that $(X^l)_{j_p^l\phi}$ does not depend on the particular choice of ϕ; this is done most readily by writing out the coordinate representation of X^l. So suppose

$$X = X^i\frac{\partial}{\partial x^i} + X^\alpha\frac{\partial}{\partial u^\alpha} \in \mathcal{X}(E).$$

Then

$$X^v = (X^\alpha - X^i u^\alpha_{1_i})\frac{\partial}{\partial u^\alpha} \in \mathcal{X}(\pi_{1,0})$$

and

$$\begin{aligned}(X^v)^l &= \sum_{|I|=0}^{l} \frac{d^{|I|}}{dx^I}(X^\alpha - X^i u^\alpha_{1_i})\frac{\partial}{\partial u^\alpha_I} \\ &= \sum_{|I|=0}^{l}\left(\frac{d^{|I|}X^\alpha}{dx^I} - \sum_{J+K=I}\frac{I!}{J!\,K!}\frac{d^{|J|}X^i}{dx^J}u^\alpha_{K+1_i}\right)\frac{\partial}{\partial u^\alpha_I} \in \mathcal{X}(\pi_{l+1,l}).\end{aligned}$$

Now since $X^i, X^\alpha \in C^\infty(E)$ and $|I|, |J| \le l$, the functions $\frac{d^{|I|}X^\alpha}{dx^I}$ and $\frac{d^{|J|}X^i}{dx^J}$ may be defined on $J^l\pi$. Therefore the only coefficients in this coordinate expansion which cannot be defined on $J^l\pi$ are those where $|I| = |K| = l$, $|J| = 0$, and the corresponding terms are

$$-X^i u^\alpha_{I+1_i}\frac{\partial}{\partial u^\alpha_I} \qquad |I| = l.$$

But now, $(j^l\phi)_*\pi_* X_{\phi(p)}$ is the tangent vector

$$X^i(\phi(p))\left(\left.\frac{\partial}{\partial x^i}\right|_{j_p^l\phi} + \sum_{|I|=0}^{l}\left.\frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}}\right|_p \left.\frac{\partial}{\partial u^\alpha_I}\right|_{j_p^l\phi}\right)$$

so that in constructing X^l one adds to $(X^v)^l$ the vector field

$$X^i\left(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{l} u^\alpha_{I+1_i}\frac{\partial}{\partial u^\alpha}\right)$$

and again the only coefficients which cannot be defined on $J^l\pi$ are those corresponding to the terms

$$X^i u^\alpha_{I+1_i}\frac{\partial}{\partial u^\alpha_I} \qquad |I| = l.$$

This shows that one may define $X^l \in \mathcal{X}(J^l\pi)$, and that its coordinate representation will be

$$X^i\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{l}\left(\frac{d^{|I|}X^\alpha}{dx^I} - \sum_{\substack{J+K=I\\ J\neq 0}}\frac{I!}{J!\,K!}\frac{d^{|J|}X^i}{dx^J}u^\alpha_{K+1_i}\right)\frac{\partial}{\partial u^\alpha_I}.$$

A similar argument shows that an identical coordinate expression holds when $X \in \mathcal{X}(\pi_{k,0})$; furthermore, either k or l (or both) may be infinite.

I can now generalise Propositions 3.2.9, 3.2.10 and 3.2.11 to vector fields which are not necessarily vertical over M. In the following two propositions I have to use a slightly clumsy pointwise description of certain vector fields; these are really just total derivatives of a certain order, lifted up to a higher jet manifold and multiplied there by functions.

Proposition 3.2.14 *If $X \in \mathcal{X}(\pi_{k,0})$ and $Y \in \mathcal{X}(M)$ then for any $m \in \mathbf{N}$ and any point $j^{k+m}_p\phi \in J^{k+m}\pi$,*

$$[Y^{k+m}, X^{m+1}]_{j^{k+m}_p\phi}$$

is the holonomic lift of a tangent vector in T_pM.

Proof This follows directly from the vanishing of $[Y^{\max\{k+m,m+1\}}, (X^v)^{m+1}]$ by Proposition 3.2.9, the construction of X^{m+1} in Definition 3.2.12 or 3.2.13, and the "closure" of holonomic lifts of vector fields under the Frölicher-Nijenhuis bracket described in Proposition 2.3.6. ∎

Proposition 3.2.15 *Suppose the vector fields $X'' \in \mathcal{X}(\pi_{k+m+1,m+1})$ and $X' \in \mathcal{X}(\pi_{k+m,m})$ are π-related. If, for every vector field $Y \in \mathcal{X}(M)$ and every point $j^{k+m}_p\phi \in J^{k+m}\pi$,*

$$[Y^{k+m}, X'']_{j^{k+m}_p\phi}$$

is the holonomic lift of a tangent vector in T_pM, then there is a vector field $X \in \mathcal{X}(\pi_{k,0})$ such that $X' = X^m$ and $X'' = X^{m+1}$.

Proof It is clear that $(X'')^v, (X')^v$ are π-related, so by the method of constructing vertical representatives and Proposition 2.3.6,

$$[Y^{k+m}, (X'')^v]_{j_p^{k+m}\phi}$$

is also the holonomic lift of a tangent vector in T_pM. But

$$[Y^{k+m}, (X'')^v]_{j_p^{k+m}\phi}(x^i) = 0$$

because $(X'')^v$ is vertical over M and Y^{k+m} is a holonomic lift, so

$$[Y^{k+m}, (X'')^v] = 0.$$

Hence by Proposition 3.2.10 $(X'')^v, (X')^v$ are the prolongations of some vertical generalised vector field X^v and the result follows by reinstating the horizontal part. ∎

The dual result is also true for contact forms.

Proposition 3.2.16 *If $X' \in \mathcal{X}(\pi_{k+m,m})$ is the prolongation of a vector field along $\pi_{k,0}$ and $\sigma \in \bigwedge^1 J^m\pi$ is a contact form then $d_{X'}\sigma \in \bigwedge^1 J^{k+m}\pi$ is a contact form. Conversely, if $X' \in \mathcal{X}(\pi_{k+m,m})$ has the property that $d_{X'}\sigma$ is a contact form on $J^{k+m}\pi$ whenever σ is a contact form on $J^m\pi$ then X' is the prolongation of a vector field along $\pi_{k,0}$.*

Proof Let $X \in \mathcal{X}(\pi_{k+m,m})$ and let X^v be its vertical representative, so that $X^v \in \mathcal{X}(\pi_{\max\{k+m,m+1\},m})$ and that $X = X^v + X^h$ (where this last equality is modulo a possible lift of X to $\mathcal{X}(\pi_{m+1,m})$). Then for any contact form σ on $J^m\pi$ I claim that $d_{X^h}\sigma = 0$: for certainly $di_{X^h}\sigma = 0$ since $X^h_{j_p^{k+m}\phi}$ is the holonomic lift of a tangent vector in M, and furthermore $d\sigma$ may be written as a combination of contact forms on $J^{m+1}\pi$ so that also $i_{X^h}d\sigma = 0$. Hence d_X preserves the module of contact forms if, and only if, d_{X^v} does; the result then follows from the method of prolonging arbitrary vector fields along $\pi_{k,0}$. ∎

I shall now use the coordinate representation of the prolongation of a vector field on E to establish an alternative method for its construction when the original vector field on E is projectable on M.

Proposition 3.2.17 *Suppose $X \in \mathcal{X}_{\overline{X}}(\pi)$, and let ψ_t be the flow of X in a neighbourhood of $\phi(p) \in E$ for given $\phi \in \Gamma_{loc}(\pi)$ and $p \in$ domain(ϕ). Then*

$$X^l_{j^l_p\phi} = [t \longmapsto \psi^l_t(j^l_p\phi)].$$

Proof Let $\overline{\psi}_t$ be the flow of $\overline{X}$ in a neighbourhood of p; then $\psi^l_t(j^l_p\phi) = j^l_{\overline{\psi}_t(p)}(\psi_t \circ \phi \circ \overline{\psi}_{-t})$. Therefore

$$\begin{aligned}
[t \longmapsto \psi^l_t(j^l_p\phi)](x^i) &= \left.\frac{d}{dt}\right|_{t=0} \left(t \longmapsto x^i(j^l_{\overline{\psi}_t(p)}(\psi_t \circ \phi \circ \overline{\psi}_{-t}))\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} (t \longmapsto x^i(\overline{\psi}_t(p))) \\
&= \left.\frac{\partial \overline{\psi}^i_t}{\partial t}\right|_{t=0} (p) \\
&= X^i(p) \\
&= X^l_{j^l_p\phi}(x^i)
\end{aligned}$$

and

$$\begin{aligned}
&[t \longmapsto \psi^l_t(j^l_p\phi)](u^\alpha_I) \\
= &\left.\frac{d}{dt}\right|_{t=0} \left(t \longmapsto u^\alpha_I(j^l_{\overline{\psi}_t(p)}(\psi_t \circ \phi \circ \overline{\psi}_{-t}))\right) \\
= &\left.\frac{d}{dt}\right|_{t=0} \left(t \longmapsto \left.\frac{\partial^{|I|}}{\partial x^I}\right|_{\overline{\psi}_t(p)} (\psi_t \circ \phi \circ \overline{\psi}_{-t})\right) \\
= &\left.\frac{\partial^{|I|}}{\partial x^I}\right|_p \left.\frac{\partial}{\partial t}\right|_{t=0} (t \longmapsto (q \longmapsto \psi^\alpha_t \circ \phi \circ \overline{\psi}_{-t}(q))) + \left.\frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}}\right|_p \left.\frac{\partial \overline{\psi}^i_t}{\partial t}\right|_{t=0} (p) \\
= &\left.\frac{\partial^{|I|}}{\partial x^I}\right|_p \left(\left.\frac{\partial \psi^\alpha_t}{\partial t}\right|_{t=0} \circ \phi + \left.\frac{\partial \phi^\alpha}{\partial x^i}\right|_p \left.\frac{\partial \overline{\psi}^i_{-t}}{\partial t}\right|_{t=0}\right) + \left.\frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}}\right|_p \left.\frac{\partial \overline{\psi}^i_t}{\partial t}\right|_{t=0} (p) \\
= &\left.\frac{\partial^{|I|}}{\partial x^I}\right|_p ((X^\alpha \circ \phi) - (u^\alpha_{1_i} \circ j^1\phi)X^i) + u^\alpha_{I+1_i} \circ j^{|I|+1}\phi(p)X^i(p) \\
= &\left.\frac{d^{|I|}X^\alpha}{dx^I}\right|_{j^{|I|}_p\phi} - \sum_{J+K=I} \frac{I!}{J!\,K!} \left.\frac{\partial^{|J|}X^i}{\partial x^J}\right|_p u^\alpha_{K+1_i}(j^{|K|+1}_p\phi) \\
&+ u^\alpha_{I+1_i}(j^{|I|+1}_p\phi)X^i(p) \\
= &\left(\frac{d^{|I|}X^\alpha}{dx^I} - \sum_{\substack{J+K=I \\ J\neq 0}} \frac{\partial^{|J|}X^i}{\partial x^J} u^\alpha_{K+1_i}\right)(j^l_p\phi) \\
= &X^l_{j^l_p\phi}(u^\alpha_I).
\end{aligned}$$

∎

In the circumstances of this proposition, X^l is often called the *complete lift* of X. Some examples:

- If π is the trivial bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$ and X is a *vertical* vector field on $\mathbf{R} \times F$ with flow $\psi_t = id_{\mathbf{R}} \times \chi_t$ then X^1 is the vector field on $J^1\pi \cong \mathbf{R} \times TF$ given by

$$X^1_{j^1_s\phi} = [t \longmapsto \psi^1_t(j^1_s\phi)] = [t \longmapsto (s, \chi_{t*}[\phi]_s)]$$

  which corresponds to the traditional definition of the complete lift of a vector field from a manifold F to its tangent manifold TF. In coordinates, if $X = X^\alpha \frac{\partial}{\partial q^\alpha}$ then

$$X^1 = X^\alpha \frac{\partial}{\partial q^\alpha} + \frac{dX^\alpha}{dt} \frac{\partial}{\partial \dot{q}^\alpha}.$$

- Now suppose that π is the trivial bundle $(M \times \mathbf{R}, pr_1, M)$ and X is a projectable vector field whose flow ψ_t is the identity on $\mathbf{R}$ in the given trivialisation, so that $\psi_t = \chi_t \times id_{\mathbf{R}}$. Then X^1 is the vector field on $J^1\pi \cong T^*M \times \mathbf{R}$ given by

$$X^1_{j^1_p\phi} = [t \longmapsto \psi^1_t(j^1_p\phi)] = [t \longmapsto ((\chi_{-t})^* d\overline{\phi}_p, \overline{\phi}(p))]$$

  which now corresponds to the traditional definition of the complete lift of a vector field from a manifold M to its cotangent manifold T^*M. Using (for this example) coordinates q^i on M, t on $\mathbf{R}$ and writing p_i for the derivative coordinates t_{1_i} on $J^1\pi$, if $X = X^i \frac{\partial}{\partial q^i}$ then

$$X^1 = X^i \frac{\partial}{\partial q^i} - p_i \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial p_j}.$$

I shall conclude this description of vector fields on jet bundles by describing the action of vertical generalised vector fields on local sections of π. The following definition is a generalisation of the corresponding property of vertical vector fields on E.

Definition 3.2.18 *If $X \in \mathcal{X}(\pi_{k,0})$ is a vertical generalised vector field and $\phi \in \Gamma_{loc}(\pi)$, define $\tilde{X}\phi \in \Gamma_{loc}(\nu_\pi)$ by*

$$\tilde{X}\phi = X \circ j^k\phi.$$

It is this action which entitles a vertical generalised vector field to be called an *evolution equation*. For suppose U is an open submanifold of M, I is an open interval in $\mathbf{R}$ and $\gamma : U \times I \longrightarrow E$ satisfies $\pi \circ \gamma(p,t) = p$ for every $(p,t) \in U \times I$. Write (as usual) γ_t for the local section of π defined by $\gamma_t(p) = \gamma(p,t)$.

Definition 3.2.19 *The map γ is called a* solution *of X if, for each $(p,s) \in U \times I$,*

$$(\tilde{X}\gamma_s)(p) = [t \longmapsto \gamma(p, s+t)].$$

In coordinates, if $X = X^\alpha \frac{\partial}{\partial u^\alpha}$ then γ is a solution of X if

$$X^\alpha(j^k_p\gamma_s)\left.\frac{\partial}{\partial u^\alpha}\right|_{\gamma_s(p)} = \left.\frac{\partial}{\partial t}\right|_{t=0}(t \longmapsto \gamma^\alpha(p,s+t))\left.\frac{\partial}{\partial u^\alpha}\right|_{\gamma_s(p)}$$

or in more traditional language, if for each α,

$$X^\alpha\left(x^i, \gamma^\beta, \frac{\partial\gamma^\beta}{\partial x^i}, \ldots, \frac{\partial^{|I|}\gamma^\beta}{\partial x^I}\right) = \frac{\partial\gamma^\alpha}{\partial t},$$

a general set of n evolution equations.

3.3 Repeated Jets

Given the bundle (E, π, M) I shall now consider the new bundle $(J^k\pi, \pi_k, M)$. The l-jet manifold of π_k is denoted $J^l\pi_k$, and contains l-jets of all the local sections of π_k:

$$J^l\pi_k = \{j^l_p\psi : \psi \in \Gamma_{loc}(\pi_k), p \in \text{domain}(\psi)\}.$$

However, suppose one restricts attention to the subset of $J^l\pi_k$ containing l-jets of those local sections which are themselves k-jet extensions of local sections of π:

$$\{j^l_p(j^k\phi) : \phi \in \Gamma_{loc}(\pi), p \in \text{domain}(\phi)\}.$$

I shall show that this subset may be canonically identified with $J^{k+l}\pi$; to do this I shall use the local coordinate system $(x^i, u^\alpha_{I;J})$, $0 \leq |I| \leq k$, $0 \leq |J| \leq l$, on $J^l\pi_k$ corresponding to the adapted local coordinate system (x^i, u^α_I), $0 \leq |I| \leq k$ on $J^k\pi$.

Proposition 3.3.1 *The map $\iota_{l,k} : J^{k+l}\pi \longrightarrow J^l\pi_k$ defined by $j^{k+l}_p\phi \longmapsto j^l_p(j^k\phi)$ is an embedding.*

Proof I shall use the coordinate relationship

$$\begin{aligned}
u^\alpha_{I;J}(j^l_p(j^k\phi)) &= \left.\frac{\partial^{|J|}}{\partial x^J}\right|_p (u^\alpha_I \circ j^k\phi) \\
&= \left.\frac{\partial^{|J|}}{\partial x^J}\right|_p \left(\frac{\partial^{|I|}\phi^\alpha}{\partial x^I}\right) \\
&= \left.\frac{\partial^{|I+J|}\phi^\alpha}{\partial x^{I+J}}\right|_p \\
&= u^\alpha_{I+J}(j^{k+l}_p\phi).
\end{aligned}$$

First, this relationship shows that $\iota_{l,k}$ is well-defined for different choices of ϕ with the same $(k+l)$-jet at p.

Secondly, it shows that $\iota_{l,k}$ is injective: for if $\iota_{l,k}(j_p^{k+l}\phi) = \iota_{k,l}(j_p^{k+l}\psi)$ then $\phi(p) = \pi_{k,0} \circ (\pi_k)_{l,0}(j_p^l(j^k\phi)) = \pi_{k,0} \circ (\pi_k)_{l,0}(j_p^l(j^k\psi)) = \psi(p)$ so that $j_p^{k+l}\phi$ and $j_p^{k+l}\psi$ are in the same coordinate patch on $J^{k+l}\pi$ with all their coordinates equal.

Thirdly, it shows that $\iota_{l,k}$ is a C^∞ immersion: for with this choice of coordinate systems $\iota_{l,k}$ corresponds to a linear map restricted to an open subset of some $\mathbf{R}^N$ and is therefore C^∞; its derivative at each point is the map itself defined on the whole of $\mathbf{R}^N$ which is injective by the argument above.

Finally, $\iota_{l,k}$ is a homeomorphism onto its image: for its inverse is continuous at each point of $\iota_{l,k}(J^{k+l}\pi)$ because a similar property holds for the corresponding map of coordinates. ∎

The following characterisation of the image of $\iota_{l,k}$ is worth having at the back of one's mind.

Lemma 3.3.2 *$\iota_{l,k}(J^{k+l}\pi)$ is the subset of $J^l\pi_k$ where, for every local coordinate system $(x^i, u^\alpha_{I,J})$, if $I_1 + J_1 = I_2 + J_2$ then $u^\alpha_{I_1;J_1}(j_p^l\psi) = u^\alpha_{I_2;J_2}(j_p^l\psi)$.*

Proof The coordinate relationship established in Proposition 3.3.1 shows that every point of $\iota_{l,k}(J^{k+l}\pi)$ satisfies the conditions of the Lemma.

Conversely, suppose a point $j_p^k\psi \in J^l\pi_k$ satisfies these conditions. Then choose a coordinate system $(x^i, u^\alpha_{I;J})$ around that point and let ϕ be a local section of π with $p \in \text{domain}(\phi)$ satisfying $u^\alpha_{I+J}(j_p^{k+l}\phi) = u^\alpha_{I;J}(j_p^l\psi)$. Then $\iota_{l,k}(j_p^{k+l}\phi) = j_p^l\psi$. ∎

Having constructed $\iota_{l,k}$ I can now use it to generalise the idea of prolonging bundle maps. Starting with the two bundles $(J^k\pi, \pi_k, M)$ and (F, ρ, N) and a bundle morphism $f : J^k\pi \longrightarrow F$ such that $\overline{f} : M \longrightarrow N$ is a diffeomorphism, I can construct the prolongation $f^l : J^{k+l}\pi \longrightarrow J^l\rho$: it is simply the composition of $\iota_{l,k}$ with the original prolongation $f^l : J^l\pi_k \longrightarrow J^l\rho$.

Some examples:

- If $\rho = \pi_k$, one sees that $\iota_{l,k}$ may be regarded as the l-th prolongation of the identity map on $J^k\pi$.

- If $M = N$, χ is a section of ρ and $R \subset J^k\pi$ is the differential equation determined by f and χ, then the l-th prolonation of R is the submanifold

$$R^l = \{j_p^{k+l}\phi : f^l(j_p^{k+l}\phi) = j_p^l\chi\} \subset J^{k+l}\pi.$$

Another interesting map is obtained by prolonging $\pi_{k,0}$ (regarded as a bundle map from π_k to π) to give the map $(\pi_{k,0})^l : J^l\pi_k \longrightarrow J^l\pi$; one finds that $(\pi_{k,0})^l(j^l_p\psi) = j^l_p(\pi_{k,0}\circ\psi)$. Notice that if $k = l$ then there are two maps from $J^k\pi_k$ to $J^k\pi$, namely $(\pi_k)_{k,0}$ and $(\pi_{k,0})^k$: this is a generalisation of the phenomenon of the two natural maps from the repeated tangent manifold TTE to TE, namely τ_{TE} (the usual tangent bundle projection) and $(\tau_E)_*$, where $\tau_E : TE \longrightarrow E$. In coordinates, $u^\alpha_I\circ(\pi_k)_{k,0} = u^\alpha_{I;}$ whereas $u^\alpha_J\circ(\pi_{k,0})^k = u^\alpha_{;J}$ so that the two maps are equal when restricted to $\iota_{k,k}(J^{2k}\pi)$.

The way all these maps fit together can be summarised in the following two commutative diagrams:

$$\begin{tikzcd}
J^{k+l+m}\pi \arrow[r, "\iota_{l+m,k}"] \arrow[d, "\pi_{k+l+m,k+l}"'] & J^{l+m}\pi_k \arrow[d, "(\pi_k)_{l+m,l}"] \\
J^{k+l}\pi \arrow[r, "\iota_{l,k}"'] & J^l\pi_k
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & J^{k+l}\pi \arrow[dl, "\pi_{k+l,l}"'] \arrow[d, "\iota_{l,k}"] \arrow[dr, "\pi_{k+l,k}"] & \\
J^l\pi \arrow[dr, "\pi_l"'] & J^l\pi_k \arrow[l, "(\pi_{k,0})^l"] \arrow[r, "(\pi_k)_{l,0}"'] \arrow[d, "(\pi_k)_l"] & J^k\pi \arrow[dl, "\pi_k"] \\
 & M &
\end{tikzcd}$$

One point about notation in this context is that in Chapter 6 I shall need to consider π_1, $(\pi_1)_1$, $((\pi_1)_1)_1$, etc. To avoid an excess of parentheses I shall adopt the following definition.

Definition 3.3.3 *The maps π_1^k are defined recursively by $\pi_1^0 = \pi$, $\pi_1^k = (\pi_1^{k-1})_1$.*

Finally in this section I shall indicate one more construction which may be performed by considering two different maps between the same pair of jet manifolds; this time the two manifolds are $J^1\pi_k$ and $J^1\pi_{k-1}$. First, the maps $(\pi_k)_{1,0} : J^1\pi_k \longrightarrow J^k\pi$ and $\iota_{1,k-1} : J^k\pi \longrightarrow J^1\pi_{k-1}$ may be composed to give the map $\iota_{1,k-1} \circ (\pi_k)_{1,0}$; secondly, the map $\pi_{k,k-1} : J^k\pi \longrightarrow J^{k-1}\pi$ may be regarded as a bundle map $\pi_k \longrightarrow \pi_{k-1}$ over the identity on M and so may be prolonged to give a map $(\pi_{k,k-1})^1$. Note that if $j_p^1\psi \in J^1\pi_k$ then its images under these two maps will be in the same fibre of $J^1\pi_{k-1}$ over $J^{k-1}\pi$: in fact

$$\begin{aligned} (\pi_{k-1})_{1,0} \circ \iota_{1,k-1} \circ (\pi_k)_{1,0}(j_p^1\psi) &= (\pi_{k-1})_{1,0} \circ (\pi_{k,k-1})^1(j_p^1\psi) \\ &= \pi_{k,k-1}(\psi(p)). \end{aligned}$$

This means that the structure of $(\pi_{k-1})_{1,0}$ as an affine bundle modelled on the vector bundle $\pi_{k-1}^*T^*M \otimes V\pi_{k-1} \longrightarrow J^{k-1}\pi$ may be used to construct the difference of these two maps.

Definition 3.3.4 *The* k-jet Spencer operator *is the map*

$$D_k : J^1\pi_k \longrightarrow \pi_{k-1}^*T^*M \otimes V\pi_{k-1}$$

defined by requiring $D_k(j_p^1\psi)$ to be the unique element of $\pi_{k-1}^*T^*M \otimes V\pi_{k-1}$ whose affine action on $J^1\pi_{k-1}$ maps $\iota_{1,k-1} \circ (\pi_k)_{1,0}(j_p^1\psi)$ to $(\pi_{k,k-1})^1(j_p^1\psi)$.

In local coordinates,

$$D_k(j_p^1\psi) = \sum_{|I|=0}^{k-1} \left(u^\alpha_{I;1_i}(j_p^1\psi) - u^\alpha_{I+1_i;}(j_p^1\psi)\right) dx^i \otimes \left.\frac{\partial}{\partial u^\alpha_I}\right|_{\pi_{k,k-1}(\psi(p))}.$$

The kernel of this operator also has a special name.

Definition 3.3.5 *The* sesquiholonomic k-jet manifold *is the submanifold $\hat{J}^{k+1}\pi \subset J^1\pi_k$ defined by*

$$\hat{J}^{k+1}\pi = D_k^{-1}(0).$$

One now has the inclusions $\iota_{1,k}(J^{k+1}\pi) \subset \hat{J}^{k+1}\pi \subset J^1\pi_k$; in terms of the coordinate characterisation given in Lemma 3.3.2 one may say that $\iota_{1,k}(J^{k+1}\pi)$ is the submanifold of $J^1\pi_k$ where the derivative coordinates are totally symmetric, whereas $\hat{J}^{k+1}\pi$ is the submanifold where all except the highest order derivative coordinates are totally symmetric (so that one may take $(x^i, u^\alpha_{I;}, u^\alpha_{J;1_i})$ as a coordinate system where $0 \leq |I| \leq k$ and $|J| = k$). This phenomenon corresponds to the fact that the highest-order derivative coordinates frequently enter calculations in a rather different way from the other coordinates, as may be seen in my discussion of contact forms in Chapter 2: see particularly Proposition 2.3.16 and also the remarks following Theorem 2.4.9. Constructions based on these ideas are used in studying the formal integrability of partial differential equations (note that when $\dim M = 1$ then $\hat{J}^{k+1}\pi$ and $\iota_{1,k}(J^{k+1}\pi)$ are identical). However I shall not consider this topic here; instead I shall content myself with the following lemma, which shows that prolonging $J^k\pi$ to $\hat{J}^{k+1}\pi$ (rather than merely to $J^{k+1}\pi$) does not introduce sections which cannot be written as jet extensions.

Lemma 3.3.6 *If $\psi \in \Gamma_{loc}(\pi_k)$ then the following are equivalent:*

1. *$\forall p \in \mathrm{domain}(\psi)$, $j^1_p\psi \in \hat{J}^{k+1}\pi$;*
2. *$\forall p \in \mathrm{domain}(\psi)$, $j^1_p\psi \in \iota_{1,k}(J^{k+1}\pi)$;*
3. *$\psi = j^k\phi$ where $\phi = \pi_{k,0} \circ \psi \in \Gamma_{loc}(\pi)$.*

Proof Suppose that, $\forall p \in \mathrm{domain}(\psi)$, $j^1_p\psi \in \hat{J}^{k+1}\pi$. I shall show that, for every contact form σ on $J^k\pi$, $\psi^*\sigma = 0$. So let $\sigma = du^\alpha_I - u^\alpha_{I+1_i}dx^i$ where $0 \leq |I| \leq k-1$. Then

$$\begin{aligned} \psi^*\sigma\big|_p &= \left(\left.\frac{\partial \psi^\alpha_I}{\partial x^i}\right|_p - \psi^\alpha_{I+1_i}(p) \right) dx^i\Big|_p \\ &= \left(u^\alpha_{I;1_i}(j^1_p\psi) - u^\alpha_{I+1_i;}(j^1_p\psi)\right) dx^i\Big|_p \\ &= 0 \end{aligned}$$

because on $\hat{J}^{k+1}\pi \subset J^1\pi_k$ the coordinates $u^\alpha_{I;1_i}$ and $u^\alpha_{I+1_i;}$ are equal when $|I| \leq k-1$. Consequently ψ is a k-jet extension by Proposition 2.3.16. The remaining implications, that if $\psi = j^k\phi$ then $j^1_p\psi \in \iota_{1,k}(J^{k+1}\pi) \subset \hat{J}^{k+1}\pi$, are immediate from Lemma 3.3.2. ∎

Chapter 4

Connections

In this chapter I shall examine the construction and use of connections in terms of jet bundles. As I have indicated in previous chapters, each jet bundle automatically provides a skeleton for the construction of connections, and an individual connection may be described by specifying its "flesh" in terms of a map from E to $J^1\pi$. This interpretation of a connection is not new, and a concise summary of the properties of such a description may be found in [50]; the same construction (given the name "slope field") may be found in a somewhat earlier description of the first-order calculus of variations [32]. I shall use the term "jet field" for the map which yields a connection, and in Section 4.1 I shall describe it in a way which emphasises the analogy with the construction of a vector field. Although traditionally a connection has been considered as a second-order object, a correct interpretation is that it is a first-order object which is frequently defined on a system which already involves derivative coordinates. In Section 4.2 I shall explain how higher-order jet fields may be described in terms of the first-order construction by using repeated jets; here, the distinction between bundles with one-dimensional base manifolds and those with higher-dimensional base manifolds will become apparent. An application of second-order jet fields to the calculus of variations will be given in Chapter 6. Finally in Section 4.3 I shall describe jet fields along maps; I shall also show how a particular type of jet field along a map may be described as a "parametrised jet field" and may be used to describe Bäcklund transformations which relate the solutions of certain partial differential equations.

Some of the material in Section 4.1 is original and will be published in [65]; Section 4.2 is an extension of a few ideas in that paper. Section 4.3 is based on the approach described in [58] and [61] but I have reinterpreted it in terms of the decomposition of a connection which I have described.

4.1 Jet Fields and Connections

I shall start this section by describing the way in which the 1-jet of a section of a bundle (E, π, M) can act on a function defined on the total space E. I shall then show how a section Γ of π_1 gives rise to a vector-valued 1-form on E, in such a way that the action of Γ on functions induced by the action of its image 1-jets is the same as the action of the vector-valued form as a derivation of type d_*. The analogy with defining the action of tangent vectors on functions, and then relating the definition of a vector field as a section of the tangent bundle to its alternative description as a derivation, should be clear. I shall then show how Γ may be combined with the horizontal operator h to produce a connection on π in the traditional sense. Finally I shall establish some properties of jet fields and their "integral sections".

Definition 4.1.1 *Given a 1-jet $j^1_p\phi \in J^1\pi$, the action of the jet on functions is the map $C^\infty(E) \longrightarrow T^*_{\phi(p)}E$ defined by*

$$j^1_p\phi[f] = \pi^*(d(\phi^* f)_p).$$

This action is well-defined for different representatives of $j^1_p\phi$ because it depends only on the first derivatives of ϕ. The main difference from the action of a tangent vector on a function is that the resulting entity is a cotangent vector lifted from the base manifold rather than a number. In coordinates, one has

$$j^1_p\phi[f] = \left(\left.\frac{\partial f}{\partial x^i}\right|_{\phi(p)} + u^\alpha_{1_i}(j^1_p\phi) \left.\frac{\partial f}{\partial u^\alpha}\right|_{\phi(p)} \right) dx^i_{\phi(p)}.$$

Definition 4.1.2 *A jet field $\Gamma : E \longrightarrow J^1\pi$ is a section of the bundle $\pi_{1,0}$. The action of Γ on functions is the map $C^\infty(E) \longrightarrow \bigwedge^1_0(\pi)$ defined by*

$$\Gamma f_{\phi(p)} = \Gamma(\phi(p))[f].$$

If the coordinate representation of Γ is given by $\Gamma^\alpha_i = u^\alpha_{1_i} \circ \Gamma$ then the action of Γ on functions may be written in coordinates as

$$\Gamma f = \left(\frac{\partial f}{\partial x^i} + \Gamma^\alpha_i \frac{\partial f}{\partial u^\alpha} \right) dx^i.$$

This action, when extended to differential forms by the rule $\Gamma(d\theta) = -d(\Gamma\theta)$, is a derivation of type d_* and suggests the following result.

Proposition 4.1.3 *There is a bijective correspondence between the jet fields $\Gamma : E \longrightarrow J^1\pi$ and the vector-valued 1-forms $R \in \bigwedge^1_0 \otimes \mathcal{X}(E)$ which satisfy the condition that $R \lrcorner\, \sigma = \sigma$ for every $\sigma \in \bigwedge^1_0 \pi$.*

Proof I shall give an explicit proof, rather than relying on the characterisation of Γ as a derivation. So suppose the jet field Γ is given. Let $a \in E$, and put $p = \pi(a)$; let ϕ be a local section of π with $p \in \mathrm{domain}(\phi)$, $a = \phi(p)$ and $j^1_p\phi = \Gamma(\phi(p))$. Define an endomorphism of tangent vectors in T_aE by $\phi_*\pi_*$ (so that the transpose of this map is the endomorphism of cotangent vectors in T^*_aE given by $\pi^*\phi^*$). This endomorphism depends only on the value and first derivatives of ϕ at p, and is therefore independent of the particular choice of ϕ satisfying the conditions given; in coordinates it may be expressed as

$$\xi^i \left.\frac{\partial}{\partial x^i}\right|_a + \eta^\alpha \left.\frac{\partial}{\partial u^\alpha}\right|_a \longmapsto \xi^i \left(\left.\frac{\partial}{\partial x^i}\right|_a + u^\alpha_{1_i}(\Gamma(a)) \left.\frac{\partial}{\partial u^\alpha}\right|_a \right).$$

Taking this endomorphism at each point $a \in E$ yields a vector-valued 1-form $\tilde{\Gamma}$ which is seen to be smooth and to satisfy the conditions of the proposition from its coordinate representation

$$\tilde{\Gamma} = dx^i \otimes \left(\frac{\partial}{\partial x^i} + \Gamma^\alpha_i \frac{\partial}{\partial u^\alpha} \right).$$

Two distinct jet fields will have different coordinate representations at some point in E so that the corresponding vector-valued forms will differ; the correspondence is therefore injective. Furthermore, any vector-valued 1-form R satisfying the conditions of the proposition must have a coordinate representation of the form

$$R = dx^i \otimes \left(\frac{\partial}{\partial x^i} + R^\alpha_i \frac{\partial}{\partial u^\alpha} \right)$$

so locally there is a jet field with coordinates R^α_i which gives rise to R; on overlapping coordinate patches these local jet fields must agree since the correspondence is injective, and so this construction defines a global jet field: the correspondence is therefore also surjective. ∎

I shall call $\tilde{\Gamma}$ the *connection* corresponding to the jet field Γ. It is clear from this result that, for a function f, $\Gamma f = d_{\tilde{\Gamma}} f$. A similar result applies to jet fields and vector-valued 1-forms defined only on a given open subset of E; in fact, one should note that the above proposition actually makes no assertion about the global existence of either type of object. Since the affine bundle $\pi_{1,0}$ only takes the additional structure of a vector bundle in favourable circumstances, the sum of two jet fields is normally undefined.

Another way of looking at this construction involves the first-order horizontal operator h, which is a vector-valued 1-form along $\pi_{1,0}$. This may be combined with a jet field Γ to produce the connection $\tilde{\Gamma}$ according to the formula

$$\tilde{\Gamma}_a(\xi) = h_{\Gamma(a)}\Gamma_*(\xi)$$

which describes the action of $\tilde{\Gamma}$ on a tangent vector $\xi \in T_aE$. Similarly, given a vector field X on M its zeroth-order holonomic lift is a vector field X^0 along $\pi_{1,0}$; using the jet field Γ one obtains a vector field $X^0 \circ \Gamma$ on the manifold E which I shall call the *horizontal lift of X* corresponding to the connection $\tilde{\Gamma}$. One may also use a jet field to act on the horizontal and vertical representatives of a vector field Y on E to give horizontal and vertical vector fields $Y^h \circ \Gamma$, $Y^v \circ \Gamma$ defined on E rather than along $\pi_{1,0}$; in fact $Y^h \circ \Gamma = Y \lrcorner \tilde{\Gamma}$, which is the traditional way of describing the horizontal component of Y relative to the connection $\tilde{\Gamma}$.

Some examples:

- If π is the trivial bundle $(M \times \mathbf{R}, pr_1, M)$ then the coordinate representation of a jet field $\Gamma : M \times \mathbf{R} \longrightarrow J^1\pi \cong T^*M \times \mathbf{R}$ is
$$\tilde{\Gamma} = dq^i \otimes \left(\frac{\partial}{\partial q^i} + \Gamma_i \frac{\partial}{\partial t}\right).$$
If I use dt to represent the pull-back $pr_2^*(dt)$ of the volume form on $\mathbf{R}$ to $M \times \mathbf{R}$, then Γ may be represented as the horizontal 1-form
$$\tilde{\Gamma} \lrcorner\, dt = \Gamma_i dq^i$$
Conversely, given a 1-form $\sigma \in \bigwedge_0^1 \pi$, σ determines a jet field by $\Gamma_\sigma(p,t) = (\sigma_p, t)$; in coordinates, if $\sigma = \sigma_i dq^i$ then
$$(\Gamma_\sigma)_i = p_i \circ \Gamma_\sigma = \sigma_i$$

- If now π is the trivial bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$ then the coordinate representation of a jet field $\Gamma : \mathbf{R} \times F \longrightarrow J^1\pi \cong \mathbf{R} \times TF$ is
$$\tilde{\Gamma} = dt \otimes \left(\frac{\partial}{\partial t} + \Gamma^\alpha \frac{\partial}{\partial q^\alpha}\right).$$
If I now consider the vector fields $Y \in \mathcal{X}_{\frac{\partial}{\partial t}}(\pi)$, then any two such vector fields Y differ by a vertical vector field, so that the contraction $Y \lrcorner \tilde{\Gamma}$ does not depend on the particular choice of Y. I may therefore write this contraction as $\frac{\partial}{\partial t} \lrcorner \tilde{\Gamma}$, and in this way obtain a representation of Γ as the "time-dependent" vector field
$$\frac{\partial}{\partial t} + \Gamma^\alpha \frac{\partial}{\partial q^\alpha}$$
Conversely, any vector field $Y \in \mathcal{X}_{\frac{\partial}{\partial t}}(\pi)$ with coordinate representation $Y = \frac{\partial}{\partial t} + Y^\alpha \frac{\partial}{\partial q^\alpha}$ will determine a vector-valued form
$$dt \otimes Y = dt \otimes \left(\frac{\partial}{\partial t} + Y^\alpha \frac{\partial}{\partial q^\alpha}\right)$$
and hence a jet field.

- Retaining π as the bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$, a jet field on the first jet bundle π_1 is a section of $(\pi_1)_1$ mapping $J^1\pi \cong \mathbf{R} \times TF \longrightarrow J^1\pi_1 \cong \mathbf{R} \times TTF$. If Γ is such a jet field with the additional property that $i_{\tilde{\Gamma}}\theta = 0$ for every $\theta \in \bigwedge_C^1(\pi)_1$ then in coordinates

$$\tilde{\Gamma} = dt \otimes \left(\frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \Gamma^\alpha \frac{\partial}{\partial \dot{q}^\alpha} \right).$$

  Γ is then an example of a *second-order jet field*, the general definition of which will be given in Section 4.2. As the base manifold of π_1 is one-dimensional, there is a representation of Γ as a vector field on $J^1\pi$:

$$\frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \Gamma^\alpha \frac{\partial}{\partial \dot{q}^\alpha}$$

  the formulation used in numerous studies of time-dependent mechanics on $\mathbf{R} \times TF$.

- If π happens to be a vector bundle then one may impose the additional requirement that the coordinate functions Γ_i^α be linear in the fibre coordinates, so that $\Gamma_i^\alpha = u^\beta \pi^*(\Gamma_{i\beta}^\alpha)$ where $\Gamma_{i\beta}^\alpha$ are functions defined locally on M. (Although I have expressed this condition in terms of coordinates, it is possible to give an intrinsic definition by introducing the notion of an affine vector field on the total space of a vector bundle.) I shall call such a map $E \longrightarrow J^1\pi$ an *affine jet field*. One may then construct the map $K : TE \longrightarrow E$ given by the composition

$$TE \longrightarrow V\pi \longrightarrow E$$

  where the first map is $I - \tilde{\Gamma}$ and the second is simply obtained from the canonical linear isomorphism of the vector spaces $V_a\pi$ and $\pi^{-1}(\pi(a))$. In coordinates, if $\eta \in T_aE$ is given by

$$\eta = \eta^i \left.\frac{\partial}{\partial x^i}\right|_a + \eta^\alpha \left.\frac{\partial}{\partial u^\alpha}\right|_a$$

  then $K\eta$ is the element of the fibre through a satisfying

$$u^\alpha(K\eta) = \eta^\alpha - \Gamma_{i\beta}^\alpha(\pi(a))u^\beta(a)\eta^i$$

  If ϕ is a section of π then the *covariant differential* of ϕ determined by the affine jet field Γ is the section $\nabla\phi$ of the tensor product bundle $T^*M \otimes E \longrightarrow M$ defined by

$$(\nabla\phi)_p(\xi) = K\phi_*\xi$$

for $\xi \in T_a M$. In coordinates, if ξ is given by $\xi^i \left.\frac{\partial}{\partial x^i}\right|_p$ then $(\nabla\phi)_p(\xi)$ is the element of the fibre $\pi^{-1}(p)$ satisfying

$$u^\alpha((\nabla\phi)_p(\xi)) = \left(\left.\frac{\partial \phi^\alpha}{\partial x^i}\right|_p - \Gamma^\alpha_{i\beta}(p)\phi^\beta(p) \right) \xi^i.$$

In the particular case when π is actually the tangent bundle (TM, τ_M, M) with coordinates (q^α) on M and $(q^\alpha, \dot{q}^\alpha)$ on TM then for a vector field $X \in \Gamma(\tau_M) = \mathcal{X}(M)$ the covariant differential ∇X is the vector-valued 1-form written in coordinates as

$$\nabla X = \left(\frac{\partial X^\alpha}{\partial q^\gamma} - \Gamma^\alpha_{\gamma\beta} X^\beta \right) dq^\gamma \otimes \frac{\partial}{\partial q^\alpha}$$

where the functions $\Gamma^\alpha_{\gamma\beta}$ are (apart from sign) the Christoffel symbls of the connection $\widetilde{\Gamma}$.

- My final example concerns the infinite jet bundle $(J^\infty\pi, \pi_\infty, M)$. A similar construction to the one I have given for the canonical inclusion map into a repeated jet manifold, $\iota_{k,l} : J^{l+k}\pi \longrightarrow J^l\pi_k$, may be used to give a map

$$\begin{aligned} \iota_{1,\infty} : J^\infty\pi &\longrightarrow J^1\pi_\infty \\ j^\infty_p\phi &\longmapsto j^1_p(j^\infty\phi) \end{aligned}$$

  which is clearly a section of $(\pi_\infty)_{1,0}$ and hence a jet field on $J^\infty\pi$. The connection determined by this jet field is just the infinite-order horizontal operator h; I have previously called this the "natural connection" on $J^\infty\pi$.

Note that if π is a trivial bundle $M \times F \longrightarrow M$ with trivial first jet bundle $M \times J^1_p\pi \longrightarrow M$ (for some $p \in M$) then it makes sense to ask whether a jet field $\Gamma : M \times F \longrightarrow M \times J^1_p\pi$ is a bundle map from $(M \times F, pr_2, F)$ to $(M \times J^1_p\pi, pr_2, J^1_p\pi)$. If it is, then one may call Γ a *base-independent jet field* and obtain an induced map $\overline{\Gamma} : F \longrightarrow J^1_p\pi$. The coordinate representation functions Γ^α_i will then be independent of x^i in any coordinate system (x^i, u^α) which respects the trivialisation $M \times F$.

An example:

- If π is the trivial bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$ then a base-independent jet field Γ gives rise to a vector field on $\mathbf{R} \times F$ written in coordinates as

$$\frac{\partial}{\partial t} + \Gamma^\alpha \frac{\partial}{\partial q^\alpha},$$

where the functions Γ^α are independent of t. This vector field is then projectable onto F in the usual sense of a projectable vector field to give

$$\Gamma^\alpha \frac{\partial}{\partial q^\alpha}$$

which of course is just the coordinate representation of the ordinary (time-independent) vector field $\overline{\Gamma} : F \longrightarrow J^1_0\pi \cong TF$.

Next I shall move on to consider integral sections of a jet field.

Definition 4.1.4 *An integral section of the jet field* Γ *is a local section* ϕ *of* π *satisfying* $j^1\phi = \Gamma \circ \phi$.

This definition clearly mimics the corresponding definition for an integral curve of a vector field. There is, however, an important difference: there is no guarantee that integral sections of a given jet field will exist, even locally. To see this, observe that each jet field Γ defines a first-order differential equation $\mathrm{Im}(\Gamma) \subset J^1\pi$ and that an integral section of Γ is a solution of this equation. (However, it should be clear that not every first-order differential equation is the image of a jet field.) In coordinates,

$$\frac{\partial \phi^\alpha}{\partial x^i} = \Gamma^\alpha_i \circ \phi$$

and this set of partial differential equations must satisfy an integrability condition if solutions ϕ^α are to exist. In fact, the following result is essentially a translation of Frobenius' theorem into the language of jet fields.

Proposition 4.1.5 *The jet field* Γ *has integral sections if, and only if, the Nijenhuis tensor of* $\widetilde{\Gamma}$ *vanishes; such a jet field is termed integrable.*

Proof From Lemma 1.4.10, if $X, Y \in \mathcal{X}(E)$ then

$$\begin{aligned} \tfrac{1}{2}(X,Y) \lrcorner N_{\widetilde{\Gamma}} &= [X,Y] \lrcorner \widetilde{\Gamma} \lrcorner \widetilde{\Gamma} + [X \lrcorner \widetilde{\Gamma}, Y \lrcorner \widetilde{\Gamma}] \\ &\quad - [X \lrcorner \widetilde{\Gamma}, Y] \lrcorner \widetilde{\Gamma} - [X, Y \lrcorner \widetilde{\Gamma}] \lrcorner \widetilde{\Gamma} \end{aligned}$$

Consider this expression locally and let X,Y be coordinate vector fields. Then $\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right) \lrcorner \widetilde{\Gamma}$ and $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u^\beta}\right) \lrcorner \widetilde{\Gamma}$ vanish identically; the only non-trivial expression comes from

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \lrcorner N_{\widetilde{\Gamma}} = \left(\left(\frac{\partial \Gamma^\alpha_j}{\partial x^i} + \Gamma^\beta_i \frac{\partial \Gamma^\alpha_j}{\partial u^\beta}\right) - \left(\frac{\partial \Gamma^\alpha_i}{\partial x^j} + \Gamma^\beta_j \frac{\partial \Gamma^\alpha_i}{\partial u^\beta}\right)\right) \frac{\partial}{\partial u^\alpha}$$

But the vanishing of this expression is precisely the condition for the equations

$$\frac{\partial \phi^\alpha}{\partial x^i} = \Gamma^\alpha_i \circ \phi$$

to be integrable in the sense of Frobenius. ∎

This condition may also be described in the language of connections. Recall that the *curvature* of a connection $\tilde{\Gamma}$ is the vector-valued 2-form $R_{\tilde{\Gamma}}$ defined by

$$(X, Y) \lrcorner R_{\tilde{\Gamma}} = [X^h \circ \Gamma, Y^h \circ \Gamma]^v \circ \Gamma$$

where I have explicitly indicated the rôle of Γ in constructing the horizontal and vertical representatives of vector fields; it is a standard result from the theory of connections that R_Γ is actually a tensor.

Proposition 4.1.6 *The jet field Γ is integrable if, and only if, the curvature $R_{\tilde{\Gamma}}$ of the associated connection $\tilde{\Gamma}$ vanishes.*

Proof I shall demonstrate the relationship between $R_{\tilde{\Gamma}}$ and $N_{\tilde{\Gamma}}$, from which the result will follow. So let $X, Y \in \mathcal{X}(E)$. If X is vertical then $X \lrcorner \tilde{\Gamma} = 0$, so

$$\begin{aligned}
\tfrac{1}{2}(X, Y) \lrcorner N_{\tilde{\Gamma}} &= [X, Y] \lrcorner \tilde{\Gamma} \lrcorner \tilde{\Gamma} - [X, Y \lrcorner \tilde{\Gamma}] \lrcorner \tilde{\Gamma} \\
&= [X, Y - Y \lrcorner \tilde{\Gamma}] \lrcorner \tilde{\Gamma} \\
&= 0
\end{aligned}$$

using $\tilde{\Gamma} \lrcorner \tilde{\Gamma} = \tilde{\Gamma}$ and the facts that $Y - Y \lrcorner \tilde{\Gamma}$ is vertical and that the bracket of two vertical vector fields is vertical. If follows from this (and the skew-symmetry of $N_{\tilde{\Gamma}}$) that $(X, Y) \lrcorner N_{\tilde{\Gamma}}$ depends only on the Γ-horizontal components of X and Y:

$$\begin{aligned}
\tfrac{1}{2}(X, Y) \lrcorner N_{\tilde{\Gamma}} &= \tfrac{1}{2}(X \lrcorner \tilde{\Gamma}, Y \lrcorner \tilde{\Gamma}) \lrcorner N_{\tilde{\Gamma}} \\
&= [X \lrcorner \tilde{\Gamma}, Y \lrcorner \tilde{\Gamma}] \lrcorner \tilde{\Gamma} \lrcorner \tilde{\Gamma} + [X \lrcorner \tilde{\Gamma} \lrcorner \tilde{\Gamma}, Y \lrcorner \tilde{\Gamma} \lrcorner \tilde{\Gamma}] \\
&\quad -[X \lrcorner \tilde{\Gamma} \lrcorner \tilde{\Gamma}, Y \lrcorner \tilde{\Gamma}] \lrcorner \tilde{\Gamma} - [X \lrcorner \tilde{\Gamma}, Y \lrcorner \tilde{\Gamma} \lrcorner \tilde{\Gamma}] \lrcorner \tilde{\Gamma} \\
&= [X \lrcorner \tilde{\Gamma}, Y \lrcorner \tilde{\Gamma}] - [X \lrcorner \tilde{\Gamma}, Y \lrcorner \tilde{\Gamma}] \lrcorner \tilde{\Gamma} \\
&= (X, Y) \lrcorner R_{\tilde{\Gamma}}
\end{aligned}$$

so that $R_{\tilde{\Gamma}} = \frac{1}{2} N_{\tilde{\Gamma}}$. ∎

The condition for the existence of integral sections may also be described in more geometric terms using Proposition 1.4.11. $\tilde{\Gamma}$ is a projection operator, and its eigenfunctions are the constant functions zero and one. The distribution corresponding to the eigenfunction zero just contains the vertical vectors, and is always involutive; its integral manifolds are the fibres of π. The distribution corresponding to the eigenfunction one may or may not be involutive; if it is, then the image sets of the integral sections will be its integral manifolds.

I should mention at this point that the natural connection h on $J^\infty\pi$ has a vanishing Nijenhuis tensor N_h, and that every local section of π_∞ which is an ∞-jet extension of a local section of π is actually an integral section of the corresponding jet field $\iota_{1,\infty}$: for by definition $\iota_{1,\infty}(j_p^\infty\phi) = j_p^1(j^\infty\phi)$ so that $\iota_{1,\infty} \circ j^\infty\phi = j^1(j^\infty\phi)$. However, on this infinite-dimensional manifold one cannot directly deduce the existence of a foliation by the images of these integral sections (at least in the C^∞ case), for given a point $j_p^\infty\phi \in J^\infty\pi$ one may always find another local section ψ of π such that $j_p^\infty\psi = j_p^\infty\phi$ but that, for some other point $q \in M$, $j_q^\infty\psi \neq j_q^\infty\phi$.

Next I shall consider symmetries of jet fields. In the case of a vector field a symmetry may be regarded as a diffeomorphism of the manifold which permutes the integral curves without changing their parametrization. It therefore seems natural to consider diffeomorphisms of E which project onto the identity of M and which permute the integral sections of Γ. So suppose ψ is such a diffeomorphism. I wish to assert that ψ is a symmetry of Γ if, whenever ϕ is an integral section, so is $\psi \circ \phi$. Using the characterisation of $\widetilde{\Gamma}_{\phi(p)}$ which was given in Proposition 4.1.3 as an endomorphism of $T_{\phi(p)}E$ for each $p \in \text{domain}(\phi)$,

$$\widetilde{\Gamma}_{\phi(p)} = \phi_*\pi_*$$

one obtains

$$\begin{aligned} \widetilde{\Gamma}_{\psi\phi(p)} &= (\psi_*\phi_*)(\pi_*\psi_*^{-1}) \\ &= \psi_*\widetilde{\Gamma}_{\phi(p)}\psi_*^{-1} \end{aligned}$$

or, more generally,

$$\widetilde{\Gamma}_{\psi(a)} = \psi_*\widetilde{\Gamma}_a\psi_*^{-1}$$

for $a \in E$. This leads to the following definition, which makes sense whether or not Γ is integrable.

Definition 4.1.7 *A* symmetry *of the jet field* Γ *is a diffeomorphism* ψ *of* E *which projects onto the identity transformation of* M *and which satisfies* $\psi_*\widetilde{\Gamma} = \widetilde{\Gamma}\psi_*$ *where* $\widetilde{\Gamma}$ *is regarded as acting on tangent vectors.*

Proposition 4.1.8 *If* Γ *is integrable then* ψ *is a symmetry of* Γ *if, and only if,* ψ *permutes the integral sections of* Γ.

∎

Corresponding to this idea is the infinitesimal version. I shall consider an infinitesimal symmetry to be a vector field whose flow consists of symmetries; as one would expect, this condition may be expressed by the vanishing of the Lie derivative. I shall establish this result in a slightly more general context.

Proposition 4.1.9 *If X is a vector field on E with flow ψ_t and R is a vector-valued 1-form on E then $d_X R = 0$ if, and only if, $\psi_{t*} \circ R = R \circ \psi_{t*}$ for each t (where R is regarded as acting on tangent vectors).*

Proof For simplicity I assume that X is complete, although this assumption is not necessary for the result to hold.

Suppose first that each ψ_t satisfies $\psi_{t*} \circ R = R \circ \psi_{t*}$. Then for every vector field Y and every point $a \in E$,

$$\begin{aligned}
(Y \lrcorner\, d_X R)_a &= (d_X(Y \lrcorner\, R))_a - (d_X Y \lrcorner\, R)_a \\
&= \left.\frac{d}{dt}\right|_{t=0} \psi_{t*}(Y \lrcorner\, R_{\psi_{-t}(a)}) - R_a\left(\left.\frac{d}{dt}\right|_{t=0} \psi_{t*} Y_{\psi_{-t}(a)}\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\psi_{t*} R_{\psi_{-t}(a)}(Y_{\psi_{-t}(a)}) - R_a(\psi_{t*} Y_{\psi_{-t}(a)})\right)
\end{aligned}$$

using continuity of the endomorphism R_a of $T_a E$. Consequently the right-hand side of this expression vanishes, and so $d_X R = 0$.

The converse involves a careful proof which effectively integrates along the flow ψ_t. So suppose that $d_X R = 0$. Then for every vector field Y and every point $a \in E$,

$$d_X(Y \lrcorner\, R)_a = R_a(d_X Y)_a$$

which I shall write as

$$\left.\frac{d}{dt}\right|_{t=0} \psi_{t*} R_{\psi_{-t}(a)}(Y_{\psi_{-t}(a)}) = \left.\frac{d}{dt}\right|_{t=0} R_a(\psi_{t*} Y_{\psi_{-t}(a)})$$

Fix a and choose an arbitrary real number h, writing $a_{-h} = \psi_{-h}(a)$; then this equation is still true with a replaced by a_{-h}. For each tangent vector $\eta \in T_{a_{-h}} E$ there is certainly a smooth vector field Y satisfying, for sufficiently small t, $Y_{\psi_{-t}(a_{-h})} = \psi_{-t*}\eta$; with this choice of Y one obtains

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \psi_{t*} R_{\psi_{-t}(a_{-h})}(\psi_{-t*}\eta) &= \left.\frac{d}{dt}\right|_{t=0} R_{a_{-h}}(\eta) \\
&= 0
\end{aligned}$$

since $R_{a_{-h}}(\eta)$ is independent of t. Also, $\eta \in T_{a_{-h}} E$ is arbitrary, so

$$\left.\frac{d}{dt}\right|_{t=0} \psi_{t*} \circ R_{\psi_{-t}(a_{-h})} \circ \psi_{-t*} = 0$$

as an endomorphism of $T_{a_{-h}} E$.

Now operate on this equation on the left by ψ_{h*} and on the right by ψ_{-h*}. The result is an equation relating endomorphisms of $T_a E$, and writing τ for $t + h$ one obtains

$$\left.\frac{d}{d\tau}\right|_{\tau=h} \psi_{\tau*} \circ R_{\psi_{-\tau}(a)} \circ \psi_{-\tau*} = 0$$

But h, too, is arbitrary and so $\psi_{\tau *} \circ R_{\psi_{-\tau}(a)} \circ \psi_{-\tau *}$ is independent of τ and therefore equals its value when τ is zero:

$$\psi_{\tau *} \circ R_{\psi_{-\tau}(a)} \circ \psi_{-\tau *} = R_a$$

so that $\psi_{\tau *} R = R \psi_{\tau *}$. ∎

I shall now apply this property to a jet field.

Definition 4.1.10 *An infinitesimal symmetry of the jet field* Γ *is a vertical vector field* X *satisfying* $d_X \tilde{\Gamma} = 0$.

Proposition 4.1.11 *If* Γ *is integrable then* X *is an infinitesimal symmetry of* Γ *if, and only if, for each* t *the diffeomorphism* ψ_t *permutes the integral sections of* Γ.

Proof This follows immediately from Proposition 4.1.9. ∎

My final remark in this section is that if the coordinate representation of the vertical vector field X is $X^\alpha \frac{\partial}{\partial u^\alpha}$, then the condition for X to be an infinitesimal symmetry of Γ takes the "Lax form"

$$\frac{\partial X^\beta}{\partial x^i} = X^\alpha \frac{\partial \Gamma_i^\beta}{\partial u^\alpha} - \Gamma_i^\alpha \frac{\partial X^\beta}{\partial u^\alpha}$$

a generalisation of the relationship

$$\frac{\partial X^\beta}{\partial t} = [X, \Gamma]^\beta$$

introduced in [47] and also discussed in the context of Lagrangian dynamics in [8,49].

4.2 Higher-order Jet Fields

All the ideas described in the previous section may be applied, in particular, to a jet bundle $(J^k\pi, \pi_k, M)$. A jet field on this bundle is then, according to Definition 4.1.2, a section Γ of the bundle $(\pi_k)_{1,0}$ and the associated connection is the vector-valued 1-form $\tilde{\Gamma}$ on $J^k\pi$ given in local coordinates by

$$\tilde{\Gamma} = dx^i \otimes \left(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} \Gamma_{I;i}^\alpha \frac{\partial}{\partial u_I^\alpha} \right)$$

where $\Gamma_{I;i}^\alpha = u_{I;1_i}^\alpha \circ \Gamma$ and $u_{I;1_i}^\alpha$ are coordinate functions on $J^1\pi_k$. However, the discussion in Section 3.3 suggests that special consideration be given to those jet fields which take their values in the submanifolds $\hat{J}^{k+1}\pi$ or $\iota_{1,k}(J^{k+1}\pi)$ of $J^1\pi_k$.

Definition 4.2.1 *A jet field* Γ *on* π_k *is called a* sesquiholonomic $(k+1)$-th order jet field on π *if* $\mathrm{Im}(\Gamma) \subset \hat{J}^{k+1}\pi$.

Definition 4.2.2 *A jet field* Γ *on* π_k *is called a* holonomic $(k+1)$-th order jet field on π*—or, more simply, just a* $(k+1)$*-th order jet field—if* $\mathrm{Im}(\Gamma) \subset \iota_{1,k}(J^{k+1}\pi)$.

I shall now establish conditions on the connection $\tilde{\Gamma}$ which correspond to the jet field Γ being a holonomic or sesquiholonomic $(k+1)$-th order jet field.

Proposition 4.2.3 *The jet field* $\Gamma : J^k\pi \longrightarrow J^1\pi_k$ *is a sesquiholonomic* $(k+1)$*-th order jet field if, and only if, for every contact form* σ *on* $J^k\pi$, $\tilde{\Gamma} \lrcorner \sigma = 0$.

Proof In local coordinates. Suppose first that $\tilde{\Gamma} \lrcorner \sigma = 0$ for each contact form σ. Take σ to be $du^\beta_J - u^\beta_{J+1_j} dx^j$ where $0 \le |J| \le k-1$; then

$$(\Gamma^\beta_{J;i} - u^\beta_{J+1_i})dx^i = 0$$

so that

$$u^\beta_{J;1_i} \circ \Gamma = u^\beta_{J+1_i}$$

demonstrating that the derivative coordinates of Γ of order $\le k$ are symmetric so that $\mathrm{Im} \subset \hat{J}^{k+1}\pi$. The argument may be reversed to show that, if Γ satisfies this condition, then $\tilde{\Gamma} \lrcorner \sigma$ vanishes for every contact form σ. ∎

In general, the condition $\tilde{\Gamma} \lrcorner \sigma = 0$ is not sufficient to ensure that Γ takes its values in $\iota_{1,k}(J^{k+1}\pi)$ because no restriction has been imposed upon the derivative coordinates of order $k+1$. To obtain a suitable condition for this purpose I shall consider the "extended" space of contact m-forms $\bigwedge^m_C(\pi)_k$.

Proposition 4.2.4 *The jet field* $\Gamma : J^k\pi \longrightarrow J^1\pi_k$ *is a holonomic* $(k+1)$*-th order jet field if, and only if, for every* $\theta \in \bigwedge^m_C(\pi)_k$, $i_{\tilde{\Gamma}}\theta = (m-1)\theta$.

Proof In local coordinates. Suppose first that $i_{\tilde{\Gamma}}\theta = (m-1)\theta$ for each $\theta \in \bigwedge^m_C(\pi)_k$. Take θ to be

$$(du^\beta_J - u^\beta_{J+1_j}dx^j) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \Omega\right)$$

where $0 \le |J| \le k-1$; then

$$(\Gamma^\beta_{J;i} - u^\beta_{J+1_i})\Omega = 0$$

so that, as in Proposition 4.2.3, $\mathrm{Im}(\Gamma) \subset \hat{J}^{k+1}\pi$. But now, taking θ instead to be

$$du^\beta_{K+1_j} \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \Omega\right) - du^\beta_{K+1_i} \wedge \left(\frac{\partial}{\partial x^j} \lrcorner \Omega\right)$$

where $|K| = k - 1$, one finds that

$$(\Gamma^{\beta}_{K+1_j;i} - \Gamma^{\beta}_{K+1_i;j})\Omega = 0$$

so that

$$u^{\beta}_{K+1_j;1_i} \circ \Gamma = u^{\beta}_{K+1_i;1_j} \circ \Gamma,$$

demonstrating that the $(k+1)$-th order derivative coordinates of Γ are also symmetric so that $\mathrm{Im}(\Gamma)$ is actually contained in $\iota_{1,k}(J^{k+1}\pi)$. It may be shown, using reasoning similar to that used in Proposition 2.3.16, that each element of $\bigwedge^m_C(\pi)_k$ may be represented in coordinates as a combination (by functions on $J^k\pi$) of the two different types of contact m-form θ described above; consequently the argument above may be reversed to show that if $\mathrm{Im}(\Gamma) \subset \iota_{1,k}(J^{k+1}\pi)$ then $i_{\tilde{\Gamma}}\theta = (m-1)\theta$ for each $\theta \in \bigwedge^m_C(\pi)_k$. ∎

From these propositions one may see that the coordinate representation of a sesquiholonomic $(k+1)$-th order jet field is

$$dx^i \otimes \left(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k-1} u^{\alpha}_{I+1_i} \frac{\partial}{\partial u^{\alpha}_I} + \sum_{|J|=k} \Gamma^{\alpha}_{J;i} \frac{\partial}{\partial u^{\alpha}_J} \right)$$

and of a holonomic $(k+1)$-th order jet field is

$$dx^i \otimes \left(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k-1} u^{\alpha}_{I+1_i} \frac{\partial}{\partial u^{\alpha}_I} + \sum_{|J|=k} \Gamma^{\alpha}_{J+1_i} \frac{\partial}{\partial u^{\alpha}_J} \right)$$

where I have used the notation $\Gamma^{\alpha}_{J+1_i}$ to indicate the symmetry of these coordinates in the holonomic case. I should remark at this point that if $m = 1$ then the conditions on $\tilde{\Gamma}$ given in these propositions are identical; however this is not very surprising because in these circumstances $\hat{J}^{k+1}\pi$ and $\iota_{1,k}(J^{k+1})$ are also identical.

I shall now consider the integral sections of higher-order jet fields. Suppose Γ is a sesquiholonomic $(k+1)$-th order jet field. From Definition 4.1.4, an integral section of Γ is a section ψ of the bundle $(J^k\pi, \pi_k, M)$ satisfying $j^1\psi = \Gamma \circ \psi$. However, the following lemma gives a more useful characterisation.

Lemma 4.2.5 *If Γ is a sesquiholonomic $(k+1)$-th order jet field then any integral section ψ of Γ is the k-jet extension of a section of π.*

Proof Let σ be a contact form on $J^k\pi$; then $\tilde{\Gamma} \lrcorner\, \sigma = 0$ so that $\pi_k^*\psi^*\sigma = 0$, and since π_k^* is injective it follows that $\psi^*\sigma = 0$. Since σ is arbitrary, Proposition 2.3.17 shows that $\psi = j^k\phi$ where $\psi = \pi_{k,0} \circ \psi$. ∎

Corollary 4.2.6 *If a sesquiholonomic $(k+1)$-th order jet field is integrable then it is actually holonomic.*

Proof If $\Gamma : J^k\pi \longrightarrow \hat{J}^{k+1}\pi$ is integrable then there is an integral section through each point of $J^k\pi$, so that each point of $\mathrm{Im}(\Gamma)$ is of the form $j^1_p(j^k\phi)$ for some local section ϕ of π and so $\mathrm{Im}(\Gamma) \subset \iota_{1,k}(J^{k+1}\pi)$. ∎

As a result of this lemma and its corollary I shall only consider holonomic $(k+1)$-th order jet fields and generally regard them as maps $J^k\pi \longrightarrow J^{k+1}\pi$ rather than $J^k\pi \longrightarrow J^1\pi_k$. I shall also regard ϕ as the integral section of such a jet field rather than $j^k\phi$. Consequently integral sections of Γ will satisfy the equations

$$\frac{\partial^{|K|}\phi^\alpha}{\partial x^K} = \Gamma^\alpha_K \circ j^k\phi \qquad |K| = k+1$$

or, in a more familiar notation,

$$\frac{\partial^{|K|}\phi^\alpha}{\partial x^K} = \Gamma^\alpha_K\left(x^k, \phi^\beta, \frac{\partial^{|I|}\phi^\beta}{\partial x^I}\right) \qquad |K| = k+1, \quad |I| \leq k.$$

4.3 Jet Fields along Bundle Maps

In the final section of this Chapter I shall show how to define jet fields along maps in an analogous way to the construction of vector fields along maps which I described in Section 1.2. Since jet fields are constructed with an explicit use of bundles, it will be necessary to consider jet fields along bundle morphisms which project onto diffeomorphisms.

Definition 4.3.1 *Let (E, π, M) and (F, ρ, N) be bundles, and let $f : \pi \longrightarrow \rho$ be a bundle morphism where $\overline{f} : M \longrightarrow N$ is a diffeomorphism. A* jet field along f *is a section of the pull-back bundle $f^*(\rho_{1,0})$.*

Just as with vector fields along maps, I shall normally regard a jet field along f as a map $\Gamma : E \longrightarrow J^1\rho$ satisfying $\rho_{1,0} \circ \Gamma = f$ rather than as a map $\Gamma : E \longrightarrow f^*(J^1\rho)$. The reason for requiring f to be a bundle morphism and for $\overline{f}$ to be a diffeomorphism arises when formulating a definition of an integral section.

Definition 4.3.2 *If Γ is a jet field along f then an* integral section of Γ *is a local section ϕ of π satisfying $\Gamma \circ \phi = j^1(\tilde{f}(\phi)) \circ \overline{f}$.*

$$\begin{array}{ccc} E & \xrightarrow{\quad\Gamma\quad} & J^1\rho \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle j^1(\tilde{f}(\phi))} \\ M & \xrightarrow[\quad\overline{f}\quad]{} & N \end{array}$$

I shall say that a jet field Γ along f is *integrable* if for each $a \in E$ there is an integral section ϕ of Γ such that $\phi(\pi(a)) = a$. In the particular case when $\pi = \rho$ and f is the identity then one just recovers the original definition of an ordinary jet field and its integral sections.

To establish a coordinate representation for a jet field along f, I shall adopt coordinates (x^i, u^α) on E and (y^a, v^A) on F: note that the ranges of the indices i and a are necessarily the same as M and N are diffeomorphic. The coordinate representation of Γ will then use the functions $\Gamma_a^A \in C^\infty(E)$, where

$$\Gamma_a^A = v_a^A \circ \Gamma.$$

Lemma 4.3.3 *If ϕ is an integral section of Γ then*

$$\Gamma_a^A \frac{\partial f^a}{\partial x^i} = \frac{\partial f^A}{\partial x^i} + \frac{\partial f^A}{\partial u^\alpha}\frac{\partial \phi^\alpha}{\partial x^i}$$

where f^a, f^A are defined by $f^a = y^a \circ f = y^a \circ \overline{f}$, $f^A = v^A \circ f$.

Proof If $p \in \operatorname{domain}(\phi)$ then

$$\begin{aligned}
\Gamma_a^A(\phi(p)) &= v_a^A \circ j^1(\tilde{f}(\phi)) \circ \overline{f}(p) \\
&= v_a^A(j^1_{\overline{f}(p)}(f \circ \phi \circ \overline{f}^{-1})) \\
&= \left.\frac{\partial}{\partial y^a}\right|_{\overline{f}(p)} (f^A \circ \phi \circ \overline{f}^{-1}) \\
&= \left(\left.\frac{\partial f^A}{\partial x^i}\right|_p + \left.\frac{\partial f^A}{\partial u^\alpha}\right|_{\phi(p)} \left.\frac{\partial \phi^\alpha}{\partial x^i}\right|_p \right) \left.\frac{\partial}{\partial y^a}\right|_{\overline{f}(p)} (\overline{f}^{-1})^i
\end{aligned}$$

so that

$$\Gamma_a^A(\phi(p))\left.\frac{\partial f^a}{\partial x^i}\right|_p = \left.\frac{\partial f^A}{\partial x^i}\right|_p + \left.\frac{\partial f^A}{\partial u^\alpha}\right|_{\phi(p)}\left.\frac{\partial \phi^\alpha}{\partial x^i}\right|_p .$$

∎

As with ordinary jet fields, a jet field along f may be represented in tensorial form; the resulting entity $\tilde{\Gamma}$ is, naturally enough, a vector-valued 1-form along f. The representation may be obtained by considering the action of the 1-jet $j^1_q\psi \in J^1\rho$ on a function $g \in C^\infty(F)$ to give the cotangent vector $\rho^*(d(\psi^*g)_q)$ at $\psi(q) \in F$, and then pulling this cotangent vector back to E using f^*. One finds that, in coordinates,

$$\tilde{\Gamma} = \frac{\partial f^a}{\partial x^i}dx^i \otimes \left(\frac{\partial}{\partial y^a} + \Gamma^A_a\frac{\partial}{\partial v^A}\right).$$

As with ordinary jet fields one may express the integrability of Γ by using $\tilde{\Gamma}$, but I shall not consider this relationship here.

Some examples:

- Let π be the trivial bundle $(\mathbf{R}\times M, pr_1, \mathbf{R})$ and ρ be the trivial bundle $(\mathbf{R}\times N, pr_1, \mathbf{R})$, and let $f : M \longrightarrow N$ so that $id_{\mathbf{R}} \times f$ is a bundle morphism from π to ρ projecting to the identity on $\mathbf{R}$. A jet field along $id_{\mathbf{R}} \times f$ is then a map $\Gamma : \mathbf{R}\times M \longrightarrow \mathbf{R}\times TN$ satisfying $\rho_{1,0}\circ\Gamma = id_{\mathbf{R}}\times f$. Taking for this example coordinates v^A on N, the representation of $\tilde{\Gamma}$ is

  $$\tilde{\Gamma} = dt \otimes \left(\frac{\partial}{\partial t} + \Gamma^A\frac{\partial}{\partial v^A}\right).$$

  As for ordinary jet bundles over a 1-dimensional base manifold, one may consider vector fields Y on $\mathbf{R}\times M$ which project to $\frac{\partial}{\partial t}$ on $\mathbf{R}$ and observe that

  $$Y \lrcorner \tilde{\Gamma} = \frac{\partial}{\partial t} + \Gamma^A\frac{\partial}{\partial v^A}$$

  is independent of the choice of Y, thus associating with Γ a "time-dependent vector field along f".

- Continuing with the same example, suppose that Γ is also a bundle map from $(\mathbf{R}\times M, pr_2, M)$ to $(\mathbf{R}\times TN, pr_2, TN)$ so that the functions Γ^A are independent of t. The induced map $\overline{\Gamma} : M \longrightarrow TN$ then satisfies $\tau_N \circ \overline{\Gamma} = f$ so that $\overline{\Gamma}$ is a vector field along f as defined in Section 1.2; in coordinates

  $$\overline{\Gamma} = \Gamma^A\frac{\partial}{\partial v^A}.$$

- Returning now to a more general case with bundles (E, π, M) and (F, ρ, M)—note that both bundles have the same base manifold M—one may consider the fibred product $\mu : J^k\pi \times_M F \longrightarrow M$ and the bundle map $pr_2 : \mu \longrightarrow \rho$ which of course projects to the identity on M. In these circumstances I shall call a jet field Γ along pr_2 a *parametrised jet field on F*; for each (global) section ϕ one obtains an ordinary jet field $\Gamma[\phi]$ on F given by

$$\Gamma[\phi](a) = \Gamma(j^k_{\rho(a)}\phi, a) \in J^1\rho$$

  where $a \in F$. The idea is that the coordinates on E (and their derivatives) enter the jet field equations as parameters.

This last example is important in the study of Bäcklund transformations, and has been examined in detail in [58,61]. I shall give a brief description of the arrangement.

Classically, a Bäcklund transformation was a system of partial differential equations with *two* sets of independent variables, u^α and v^A. If u^α were regarded as parameters then the integrability conditions on the derivatives of v^A yielded some partial differential equations which u^α had to satisfy; the opposite point of view yielded some more partial differential equations, this time for v^A. The original example studied by Bäcklund concerned the sine-Gordon equation: the transformation equations were

$$\begin{aligned} u_x - v_x &= 2\lambda \sin \tfrac{1}{2}(u + v) \\ u_t + v_t &= 2\lambda^{-1} \sin \tfrac{1}{2}(u - v) \end{aligned}$$

where u, v were both functions of x and t. By substituting a particular function v so that $v_{xt} = v_{tx}$ one obtains

$$\begin{aligned} 2u_{xt} &= 2\lambda \cos \tfrac{1}{2}(u + v).\tfrac{1}{2}(u_t + v_t) + 2\lambda^{-1} \cos \tfrac{1}{2}(u - v).\tfrac{1}{2}(u_x - v_x) \\ &= 2 \cos \tfrac{1}{2}(u + v) \sin \tfrac{1}{2}(u - v) + 2 \cos \tfrac{1}{2}(u - v) \sin \tfrac{1}{2}(u + v) \\ &= 2 \sin u, \end{aligned}$$

and similarly substituting a particular function u so that $u_{xt} = u_{tx}$ one obtains $v_{xt} = \sin v$. In this instance the equations corresponding to the two integrability conditions are the same (the sine-Gordon equation in light-cone coordinates) and the transformation is termed an auto-Bäcklund transformation. When $\lambda = 1$ this is sometimes called the Bianchi transformation and it has an interesting geometrical interpretation in terms of surfaces of constant negative Gaussian curvature embedded in $\mathbf{R}^3$ [24].

An example where the two equations for u and v are different is

$$\begin{aligned} v_x &= -2u - v^2 \\ v_t &= 8u^2 + 4uv^2 + 2u_{xx} - 4u_x v. \end{aligned}$$

Substituting a particular function v so that $v_{xt} = v_{tx}$ one obtains the equation

$$u_{xxx} + 12uu_x + u_t = 0.$$

On the other hand, taking the v_x equation as defining u in terms of v and substituting into the v_t equation gives

$$v_{xxx} - 6v^2 v_x + v_t = 0.$$

These equations for u and v, or certain variants with different numerical coefficients, are known as the Korteweg-de Vries and Modified Korteweg-de Vries equations respectively, and they will appear again in subsequent chapters. The transformation between them is therefore an example of a Bäcklund transformation which is not an auto-Bäcklund transformation [68].

This latter example shows that the correspondence between the two equations (or systems of equations) related by a Bäcklund transformation is not necessarily symmetric, and this is reflected in the formulation in terms of jet fields. The parametrised jet field $\Gamma : J^k\pi \times_M F \longrightarrow J^1\rho$ is termed *integrable with respect to* ϕ (where ϕ is a section of π) if the ordinary jet field $\Gamma[\phi]$ is integrable; the integrability conditions on $\Gamma[\phi]$ involve the first derivatives of its coefficient functions and therefore become a system of $(k+1)$-th order differential equations which ϕ must satisfy. If these equations do not involve the variables v^A then they determine a fibred submanifold S of $J^{k+1}\pi$; it may also be the case that S is defined by a differential operator so that prolongations of the differential operator will define prolongations of S. There may then exist some $r \in \mathbf{N}$ such that the prolongation

$$\Gamma^r(S^{r-1} \times_M F)$$

is a fibred submanifold of $J^r\rho$; Γ is then called an ordinary Bäcklund map and the two systems of equations represented by the fibred submanifolds of $J^{k+1}\pi$ and $J^r\rho$ are said to be related by a Bäcklund transformation. As the details of this construction may be found in [58] I shall not repeat them here; I shall, however, return to this description of a Bäcklund map in Section 7.2.

Chapter 5

A Generalised Vertical Endomorphism

The vector-valued 1-form v described at the end of Chapter 2 may be used to describe vectors on a jet manifold which are vertical over the base manifold M. My objective in the present chapter is to introduce a new geometrical object which reflects the fact that vectors may also be vertical over the total space E. The prototype of this object is the almost tangent structure (or "vertical endomorphism", or "S tensor") on a tangent manifold, and so in Section 5.1 I review this construction and describe some of its applications (principally to Lagrangian mechanics) which have appeared in the recent literature. I also refer to a similar object which has been defined on higher-order tangent manifolds.

In the remaining three sections of this chapter I show how this construction may be generalised to the more complicated environment of a jet manifold, where the base manifold M need not be 1-dimensional. This material is original, and the essential elements of it have been published [63]. The present exposition differs from the published work in that I have adopted a slightly different notation, added more details in certain proofs, and included a considerable amount of extra information about the properties of the various operators.

5.1 The Original S Tensor

My intention in this section is to explain the construction of the "original S tensor" on a tangent manifold and briefly to justify its alternative description as an almost tangent structure. I shall mention some of the applications of this construction, and I shall also show how a version of the S tensor can be constructed on a higher-order tangent manifold.

To put these ideas into context, recall that on a manifold E one may define various types of geometrical structures: for example, a non-degenerate type (2,0) tensor field defines a (pseudo)-Riemannian geometry; a closed, non-degenerate 2-form defines a symplectic geometry; a vector-valued 1-form J satisfying $J \lrcorner J = -I$ and $N_J = 0$ defines an almost complex geometry. The last of these examples is particularly relevant because of the way that the vector-valued form J characterises the geometrical structure, so I shall describe it in a little more detail.

Suppose for a moment that $E_{\mathbf{C}}$ is a *complex* manifold of complex dimension n (that is, whose coordinate mappings take their values in $\mathbf{C}^n$ and whose transition functions are complex holomorphic functions). Then at any point $a \in E_{\mathbf{C}}$ the tangent space $T_a E_{\mathbf{C}}$ is a complex vector space and so has an endomorphism J_a given by $J_a : \xi \longmapsto i\xi$. $E_{\mathbf{C}}$ may be regarded as a smooth (in fact, real-analytic) $2n$-dimensional manifold E, and then at each point $a \in E$ there is correspondingly an endomorphism J_a of $T_a E$. The map $J : E \longrightarrow T^*E \otimes TE$ defined in this way is a vector-valued 1-form clearly satisfying $J \lrcorner J = -I$, and a simple calculation shows that $N_J = 0$. Conversely, suppose that E is an ordinary smooth real manifold of dimension $2n$ and that a vector-valued 1-form with these properties has been specified. Then a deep theorem of Newlander and Nirenberg [54] asserts that it is possible to construct $\mathbf{C}^n$-valued coordinate maps on E such that, when $\mathbf{C}^n$ is viewed as $\mathbf{R}^{2n}$, these charts are contained in the maximal atlas of smooth $\mathbf{R}^{2n}$-valued charts defined by the original manifold structure of E; furthermore, the vector-valued 1-form J then corresponds to multiplication by i in the tangent spaces. In this sense J may be said to characterise those properties of a manifold which arise when it is a complex manifold. J is called an *integrable almost complex structure*; if the Nijenhuis condition $N_J = 0$ is not known to hold then J is merely termed an *almost complex structure*.

The analogy with the S tensor on a tangent manifold is close but not complete. If (E, π, M) is a vector bundle then for each $a \in E$ the fibre of π through a is canonically isomorphic, as a vector space, to the vertical tangent space $V_a\pi$: this is because $V_a\pi$ contains just those tangent vectors at a which are actually tangent to the fibre, so that $V_a\pi = T_a\pi^{-1}(\pi(a)) \cong \pi^{-1}(\pi(a))$ using the canonical isomorphism of a vector space with its tangent space at any point. To be specific, if $\xi \in \pi^{-1}(\pi(a))$ then the image of ξ under this isomorphism is $[t \longmapsto a + t\xi] \in V_a\pi$. In the particular case of a tangent bundle (TM, τ_M, M) this isomorphism becomes a map v_a from $T_{\tau_M(a)}$ to $V_a\tau_M$ which may be written in coordinates as

$$\xi^i \left.\frac{\partial}{\partial q^i}\right|_{\tau_M(a)} \longmapsto \xi^i \left.\frac{\partial}{\partial \dot{q}^i}\right|_a .$$

Definition 5.1.1 *The* vertical endomorphism on TM *is the vector-valued 1-form* $S : TM \longrightarrow T^*TM \otimes TTM$ *defined by its pointwise action on tangent vectors:*

$$S_a = \mathrm{v}_a \circ \tau_{M*} : T_aTM \longrightarrow V_a\tau_M \subset T_aTM.$$

This action may be written in coordinates as

$$\xi^i \left.\frac{\partial}{\partial q^i}\right|_a + \eta^i \left.\frac{\partial}{\partial \dot{q}^i}\right|_a \longmapsto \xi^i \left.\frac{\partial}{\partial \dot{q}^i}\right|_a$$

and so

$$S = dq^i \otimes \frac{\partial}{\partial \dot{q}^i}.$$

Lemma 5.1.2 *S has the following properties:*

1. *S has constant rank $m = \dim M$;*
2. *$S \lrcorner S = 0$;*
3. *$N_S = 0$.*

Proof

1. $\operatorname{rank} \tau_{M*} = m$ and v_a is an isomorphism;
2. $\tau_{M*} \circ \mathrm{v}_a = 0$;
3. simplest in coordinates, because $\frac{\partial}{\partial q^i} \lrcorner S = \frac{\partial}{\partial \dot{q}^i}$ and $\frac{\partial}{\partial \dot{q}^i} \lrcorner S = 0$ so that all the brackets in the expression for N_S vanish when acting on the coordinate vector fields; since N_S is a tensor this implies that it vanishes.

∎

Any vector-valued 1-form S on a $2m$-dimensional manifold E with the properties listed in Lemma 5.1.2 is called an *integrable almost tangent structure*; if the condition $N_S = 0$ is not known to hold then S is referred to simply as an *almost tangent structure*. The lemma therefore shows that the vertical endomorphism on a tangent manifold is an integrable almost tangent structure.

To continue the analogy with almost complex structures, one might next enquire whether the existence of an integrable almost tangent structure S on a manifold E implied that E may be identified with a suitable tangent manifold. Certainly the existence of S imposes certain topological restrictions on E (for example, that it is orientable). Furthermore, the kernel of S is an m-dimensional distribution which is equal to the image of S; from the integrability condition $N_S = 0$ one obtains

$$[X \lrcorner S, Y \lrcorner S] = [X \lrcorner S, Y] \lrcorner S + [X, Y \lrcorner S] \lrcorner S$$

so that this distribution is involutive and therefore has integral manifolds which define a foliation of E. (I have used a special argument here because this result does *not* follow from Proposition 1.4.11 since S is not diagonalisable.) However this is not sufficient to ensure that E is a tangent manifold. For example, consider the torus $S^1 \times S^1$: for each a, $T_a(S^1 \times S^1) \cong T_{pr_1(a)}S^1 \oplus T_{pr_2(a)}S^1$ so define an endomorphism $(\xi_1, \xi_2) \longmapsto (0, \xi_1)$ where the metric structure on S^1 has implicitly been used to identify different tangent spaces. The corresponding vector-valued 1-form satisfies the conditions of Lemma 5.1.2 but of course the integral manifolds of its kernel distribution are just the circles $\{pr_1(a)\} \times S^1$ which cannot themselves be tangent spaces.

The earliest reference I can find to integrable almost tangent structures is in [7]. The question of whether a manifold with such a structure is a tangent manifold was considered in [5]; I shall henceforth only be concerned with the canonical integrable almost tangent structure on TM. This operator S has now appeared in numerous works on dynamical systems. The relationship of S to the horizontal distribution determined by a second-order differential equation (and, in particular, by a spray) was considered in [9]. Connections X on a tangent bundle with its zero section deleted, and which satisfy the conditions $X \lrcorner S = -S$, $S \lrcorner X = S$ were studied in [33,34]. Several applications of S to Lagrangian dynamics were examined in [11], and the operator has also been used in the consideration of bi-Lagrangian systems where two Lagrangian functions which are neither mutiples of each other nor differ by a total time derivative nevertheless have the same variational equations [4,13,26,27].

This interest in the use of S has prompted generalisations to higher-order tangent manifolds, so that similar techniques may be applied to higher-order dynamical systems. The object of interest is now a vector-valued 1-form on T^kM. This cannot be constructed by a straightforward generalisation of the technique used for TM because $\tau_M^k : T^kM \longrightarrow M$ is not a vector bundle, and so its fibres are not vector spaces. I shall mention two approaches, one with an algebraic flavour and one which is more analytic.

A description of the algebraic approach may be found in [17]. One starts with the two exact sequences of vector bundles over T^kM corresponding to its fibrations over $T^{k-1}M$ and M respectively:

$$0 \longrightarrow V\tau_M^{k,k-1} \longrightarrow TT^kM \longrightarrow \tau_M^{k,k-1*}T^{k-1}M \longrightarrow 0$$

$$0 \longrightarrow V\tau_M^k \longrightarrow TT^kM \longrightarrow \tau_M^{k*}TM \longrightarrow 0$$

where $\tau_M^{k,k-1} : T^kM \longrightarrow T^{k-1}M$ is the natural projection (entirely analogous to $\pi_{k,k-1}$ in the context of jet bundles). There is then a vector bundle isomorphism from the transverse bundle of the upper sequence to the vertical bundle of the lower sequence. In terms of fibre coordinates this map is given by

$$q^i_{(r+1)}(h(\xi)) = q^i_{(r)}(\xi) \qquad 0 \le r \le k-1$$

but is independent of the choice of coordinate system. By composition one then obtains a vector bundle map $TT^kM \longrightarrow TT^kM$. (Of course, coordinates could also have been used to define this map directly.)

The second approach may be found in [15]. In this approach a point in T^kM is represented by an equivalence class of functions $\mathbf{R} \longrightarrow M$, so that a tangent vector at that point is an equivalence class of functions $\mathbf{R}^2 \longrightarrow M$ where the new equivalence relation involves derivatives of order up to k in the first variable but no more than first order in the second variable. An operation of vertical lift from $T_{\tau_M^{k,k-1}(a)}T^{k-1}M$ to $V_a\tau_m^k$ is then specified by representing a tangent vector ξ using a function $\gamma : \mathbf{R}^2 \longrightarrow M$ and then defining the vertical lift of ξ using the new function $(s,t) \longmapsto \gamma(s, st)$.

Using either approach, one obtains a vector-valued 1-form S on T^kM which has the following properties:

1. S has constant rank km;
2. $S^{k+1} = 0$;
3. $N_S = 0$.

and in coordinates may be written as

$$S = \sum_{r=0}^{k-1}(r+1)dq^i_{(r)} \otimes \frac{\partial}{\partial q^i_{(r+1)}}.$$

It is clear that this higher-order vertical endomorphism is a generalisation of the original S on TM, although not all of the properties carry over. Interestingly enough, the reason for this will become apparent when I consider the extension of these ideas to jet bundles.

5.2 The Vertical Endomorphism defined by a 1-form

In this section I shall return to my usual notation where (E, π, M) is a bundle. My intention is to show how a "vertical endomorphism" may be defined on jet manifolds $J^k\pi$ in a way which generalises the definition for tangent manifolds. The endomorphism is to have the property that, at any point of $J^k\pi$, the image of a tangent vector should be another tangent vector which is vertical over E; furthermore, when the base manifold M is one-dimensional the resulting object should bear a close relationship to the original vertical endomorphism described in the previous section.

This construction can certainly be carried out in an intrinsic manner, using a combination of a projection and a vertical lift, and it yields an endomorphism with the required properties. However, when $\dim M > 1$ it is necessary to make a "choice of direction on M" and consequently one obtains a family of vertical endomorphisms rather than a unique operator. (In Section 5.4 I shall show how—on $J^1\pi$—these may all be combined in a single object, and when I come to consider the Cartan form in Chapter 6 I shall indicate how a similar process may be carried out on $J^k\pi$ ($k > 1$) [63].) The reason for this "choice of direction" is clear when one attempts to write down the effect of such an endomorphism in coordinates:

$$\text{on } TE, \qquad \frac{\partial}{\partial q^\alpha} \longmapsto \frac{\partial}{\partial \dot{q}^\alpha}$$
$$\text{on } J^1\pi, \qquad \frac{\partial}{\partial u^\alpha} \overset{?}{\longmapsto} \frac{\partial}{\partial u^\alpha_{1_i}}.$$

There are clearly too many candidates for the image of $\frac{\partial}{\partial u^\alpha}$, and the position of the unbalanced index suggests that the input data should include a 1-form on M:

$$\left(dx^i, \frac{\partial}{\partial u^\alpha}\right) \overset{?}{\longmapsto} \frac{\partial}{\partial u^\alpha_{1_i}}.$$

This is indeed the case, although the operation is not as straightforward as one might imagine: while on $J^1\pi$ it is certainly true that

$$\left(\omega_i dx^i, \xi^\alpha \frac{\partial}{\partial u^\alpha}\right) \longmapsto \xi^\alpha \omega_i \frac{\partial}{\partial u^\alpha_{1_i}}$$

on $J^2\pi$ one has instead

$$\left(\omega_i dx^i, \xi^\alpha \frac{\partial}{\partial u^\alpha}\right) \longmapsto \xi^\alpha \left(\omega_i \frac{\partial}{\partial u^\alpha_{1_i}} + \frac{\partial \omega_i}{\partial x^j} \frac{\partial}{\partial u^\alpha_{1_i+1_j}}\right)$$

and in general the operator on higher jet manifolds involves derivatives of the coefficients of the 1-form. This is an unfortunate complication, but it is necessary for the operator to behave properly under coordinate changes and it arises naturally in the construction I shall describe.

Definition 5.2.1 *Given a point $j^k_p\phi \in J^k\pi$, a closed 1-form ω defined in a neighbourhood of $p \in M$ and a tangent vector $\xi \in V_{j^{k-1}_p\phi}(\pi_{k-1})$, the* vertical lift of ξ by ω to $j^k_p\phi$ *corresponding to an adapted coordinate system (x^i, u^α_I) around $j^k_p\phi$ is the tangent vector $\xi \otimes_{j^k_p\phi} \omega$ specified as follows: if*

$$\xi = \sum_{|I|=0}^{k-1} \xi^\alpha_I \left.\frac{\partial}{\partial u^\alpha_I}\right|_{j^{k-1}_p\phi} \qquad \textit{and} \quad \omega = \omega_i dx^i$$

then

$$\xi \circledvee_{j^k_p\phi} \omega = \sum_{|J+K|=0}^{k-1} \frac{(J+K+1_i)!}{(J+1_i)!\,K!} \xi^\alpha_K \left. \frac{\partial^{|J|}\omega_i}{\partial x^J} \right|_p \left. \frac{\partial}{\partial u^\alpha_{J+K+1_i}} \right|_{j^k_p\phi}.$$

My notation does not indicate dependence of the vertical lift on the choice of coordinate system; this omission is justified by the following result.

Theorem 5.2.2 *$\xi \circledvee_{j^k_p\phi} \omega$ may be defined intrinsically and is therefore independent of a choice of coordinates.*

Proof First, since the 1-form ω is closed, there is a function f defined in a neighbourhood of $p \in M$ satisfying $df = \omega$; the germ of f is unique to within an additive constant which I shall specify by requiring $f(p) = 0$.

Next I consider the maps $\gamma : U \times \mathbf{R} \longrightarrow E$, where U is some neighbourhood of p, which satisfy (defining $\gamma_t : U \longrightarrow E$ by $\gamma_t(q) = \gamma(q,t)$)

$$\gamma_t \in \Gamma_{loc}(\pi), \quad [t \longmapsto j^{k-1}_p \gamma_t] = \xi.$$

As in Proposition 3.2.1, each vector ξ vertical over M may be defined by such a map γ, and it is certainly possible (by adding a suitable polynomial function to the coordinate representation) to choose such a γ which also satisfies $j^k_p\gamma_0 = j^k_p\phi$. I can then use the map γ and the function f to construct a new map $\chi : U \times \mathbf{R} \longrightarrow E$ by the rule

$$\chi(q,t) = \gamma(q, tf(q)).$$

Defining $\chi_t : U \longrightarrow E$ by $\chi_t(q) = \chi(q,t)$, one finds that

$$j^k_p\chi_0 = j^k_p\gamma_0 = j^k_p\phi$$

so that

$$[t \longmapsto j^k_p\chi_t]$$

is a tangent vector in $T_{j^k_p\phi}(J^k\pi)$ which I shall also denote $\xi \circledvee_{j^k_p\phi} \omega$ in anticipation of my proof that it has the coordinate representation given in Definition 5.2.1.

Now

$$\left. \frac{\partial \chi^\alpha}{\partial t} \right|_{t=0} (q) = f(q) \left. \frac{\partial \gamma^\alpha}{\partial t} \right|_{t=0} (q) \quad (q \in U)$$

and so, by the higher-order version of Leibniz' rule (Proposition 2.1.3)

$$\left. \frac{\partial^{|I|+1}\chi^\alpha}{\partial t\,\partial x^I} \right|_{t=0;p} = \sum_{J+K=I} \frac{I!}{J!\,K!} \left. \frac{\partial^{|J|} f}{\partial x^J} \right|_p \left. \frac{\partial^{|K|+1}\gamma^\alpha}{\partial t\,\partial x^K} \right|_{t=0;p}$$

demonstrating that $\xi \bigotimes_{j^k_p\phi} \omega$ depends on the vector ξ rather than the particular map γ chosen to represent it, and in particular that

$$\left.\frac{\partial \chi^\alpha}{\partial t}\right|_{t=0;p} = f(p) \left.\frac{\partial \gamma^\alpha}{\partial t}\right|_{t=0;p} = 0$$

so that $\xi \bigotimes_{j^k_p\phi} \omega$ is vertical over E. Finally

$$\begin{aligned}
\xi \bigotimes_{j^k_p\phi} \omega &= \sum_{|I|=0}^{k} \left.\frac{\partial^{|I|+1}\chi^\alpha}{\partial t\, \partial x^I}\right|_{t=0;p} \left.\frac{\partial}{\partial u^\alpha_I}\right|_{j^k_p\phi} \\
&= \sum_{|I|=0}^{k} \sum_{J+K=I} \frac{I!}{J!\,K!} \left.\frac{\partial^{|K|+1}\gamma^\alpha}{\partial t\, \partial x^K}\right|_{t=0;p} \left.\frac{\partial^{|J|} f}{\partial x^J}\right|_p \left.\frac{\partial}{\partial u^\alpha_I}\right|_{j^k_p\phi} \\
&= \sum_{|J+K|=0}^{k} \frac{(J+K)!}{J!\,K!} \left.\frac{\partial^{|K|+1}\gamma^\alpha}{\partial t\, \partial x^K}\right|_{t=0;p} \left.\frac{\partial^{|J|} f}{\partial x^J}\right|_p \left.\frac{\partial}{\partial u^\alpha_I}\right|_{j^k_p\phi} \\
&= \sum_{|J+K|=0}^{k-1} \frac{(J+K+1_i)!}{(J+1_i)!\,K!} \left.\frac{\partial^{|K|+1}\gamma^\alpha}{\partial t\, \partial x^K}\right|_{t=0;p} \left.\frac{\partial^{|J|+1_i} f}{\partial x^{J+1_i}}\right|_p \left.\frac{\partial}{\partial u^\alpha_{J+K+1_i}}\right|_{j^k_p\phi} \\
&= \sum_{|J+K|=0}^{k-1} \frac{(J+K+1_i)!}{(J+1_i)!\,K!} \xi^\alpha_K \left.\frac{\partial^{|J|}\omega_i}{\partial x^J}\right|_p \left.\frac{\partial}{\partial u^\alpha_{J+K+1_i}}\right|_{j^k_p\phi}.
\end{aligned}$$

∎

This construction applies equally well to the case $k = \infty$ if one replaces $\infty - 1$ by ∞; notice that the coordinate representation of $\xi \bigotimes_{j^\infty_p\phi} \omega$ still makes sense because each coefficient is the sum of only a finite number of terms.

A reassuring property of vertical lifts is that they are compatible with the jet bundle structure.

Lemma 5.2.3 *If* $\xi \in V_{j^{k-1}_p\phi}(\pi_{k-1})$ *and* $k > l > 0$ *then* $\pi_{k,l*}(\xi \bigotimes_{j^k_p\phi} \omega) = (\pi_{k-1,l-1*}\xi) \bigotimes_{j^l_p\phi} \omega$.

Proof Using the notation of Theorem 5.2.2,

$$\pi_{k-1,l-1*}\xi = [t \longmapsto j^{l-1}_p \gamma_t]$$

so that

$$\begin{aligned}
(\pi_{k-1,l-1*}\xi) \bigotimes_{j^l_p\phi} \omega &= [t \longmapsto j^l_p \chi_t] \\
&= \pi_{k,l*}[t \longmapsto j^k_p \chi_t] \\
&= \pi_{k,l*}(\xi \bigotimes_{j^k_p\phi} \omega).
\end{aligned}$$

∎

I can now apply this operation to vector fields defined on $J^{k-1}\pi$ and vertical over M, to give new vector fields defined on $J^k\pi$ and vertical over E.

Definition 5.2.4 *If $\omega \in \bigwedge^1 M$ with $d\omega = 0$, and $X \in \mathcal{V}(\pi_{k-1})$, then the* vertical lift of X by ω *is the vector field $X \circledvee \omega \in \mathcal{V}(\pi_{k,0})$ defined by*

$$(X \circledvee \omega)_{j_p^k\phi} = X_{j_p^{k-1}\phi} \circledvee_{j_p^k\phi} \omega.$$

I shall not normally indicate the particular map $\pi_{k,k-1}$ along which the vertical lift $\circledvee$ operates. In coordinates, if $X = \sum_{|I|=0}^{k-1} X_I^\alpha \frac{\partial}{\partial u_I^\alpha}$ and $\omega = \omega_i dx^i$ then

$$X \circledvee \omega = \sum_{|J+K|=0}^{k-1} \frac{(J+K+1_i)!}{(J+1_i)!\,K!} X_K^\alpha \frac{\partial^{|J|}\omega_i}{\partial x^J} \frac{\partial}{\partial u_{J+K+1_i}^\alpha}.$$

Now suppose that ω^1, ω^2 are two closed 1-forms on M. The operation of vertical lift may therefore be performed twice, from $J^{k-2}\pi$ to $J^k\pi$; I shall show that the order in which these operations are carried out is immaterial.

Lemma 5.2.5 *If $\omega^1, \omega^2 \in \bigwedge^1 M$ with $d\omega^1 = d\omega^2 = 0$, and $X \in \mathcal{V}(\pi_{k-2})$, then*

$$X \circledvee \omega^1 \circledvee \omega^2 = X \circledvee \omega^2 \circledvee \omega^1.$$

Proof Let $j_p^{k-2}\phi \in J^{k-2}\pi$, and write ξ for $X_{j_p^{k-2}\phi}$. In Theorem 5.2.1, let ξ be represented by a map γ which still satisfies $j_p^k\gamma_0 = j_p^k\phi$, even though now

$$\xi = [t \longmapsto j_p^{k-2}\gamma_t].$$

If locally $\omega^1 = df^1$ and $\omega^2 = df^2$ then

$$\xi \circledvee_{j_p^{k-1}\phi} \omega^1 = [t \longmapsto j_p^{k-1}(q \longmapsto \gamma(q, tf^1(q)))]$$

and from the additional restriction on γ, one obtains

$$(\xi \circledvee_{j_p^{k-1}\phi} \omega^1) \circledvee_{j_p^k\phi} \omega^2 = [t \longmapsto j_p^k(q \longmapsto \gamma(q, tf^1(q)f^2(q)))]$$

which clearly is also equal to $(\xi \circledvee_{j_p^{k-1}\phi} \omega^2) \circledvee_{j_p^k\phi} \omega^1$. ∎

As a result of this lemma I can use a multi-index notation for repeated vertical lifts. So suppose that $(\omega^1, \ldots, \omega^m)$ is a family of m closed 1-forms on M, and that $X \in \mathcal{V}(\pi_l)$. For any multi-index $I \in \mathbf{N}^m$, define the repeated lift $X \circledvee \omega^I$ recursively:

$$X \circledvee \omega^{I+1_i} = X \circledvee \omega^I \circledvee \omega^i.$$

Of course, the idea is that $(\omega^1, \ldots, \omega^m)$ should form a basis of closed 1-forms: however, the topological nature of M may prohibit this, so my definition will also allow the use of a family which only forms a basis on some open submanifold of M.

I shall now combine the operation of the vertical lift of tangent vectors with the vertical vector-valued form along $\pi_{k,k-1}$ to define a vertical endomorphism.

Definition 5.2.6 *If* $\omega \in \bigwedge^1 M$ *with* $d\omega = 0$, *then the* vertical endomorphism of order k defined by ω *is the vector-valued 1-form* $S_\omega^{(k)}$ *defined by its action on a tangent vector* $\xi \in T_{j_p^k\phi}(J^k\pi)$:

$$(S_\omega^{(k)})_{j_p^k\phi}(\xi) = v_{j_p^k\phi}(\xi) \circledvee_{j_p^k\phi} \omega.$$

In coordinates,

$$S_\omega^{(k)} = \sum_{|J+K|=0}^{k-1} \frac{(J+K+1_i)!}{(J+1_i)!\,K!} \frac{\partial^{|J|}\omega_i}{\partial x^J}(du_K^\alpha - u_{K+1_j}^\alpha dx^j) \otimes \frac{\partial}{\partial u_{J+K+1_i}^\alpha}.$$

For a given 1-form ω, the vertical endomorphisms on different jet manifolds are compatible with the bundle structure as a consequence of Lemma 5.2.3.

Lemma 5.2.7 *If* $X \in \mathcal{X}(J^k\pi)$ *is* $\pi_{k,l}$*-related to* $Y \in \mathcal{X}(J^l\pi)$ *then* $X \lrcorner S_\omega^{(k)}$ *is* $\pi_{k,l}$*-related to* $Y \lrcorner S_\omega^{(l)}$. *If* $\sigma \in \bigwedge^1 J^l\pi$ *then* $S_\omega^{(k)} \lrcorner \pi_{k,l}^*\sigma = \pi_{k,l}^*(S_\omega^{(l)} \lrcorner \sigma)$.

Proof For each $j_p^k\phi \in J^k\pi$,

$$\begin{aligned}
\pi_{k,l*}((S_\omega^{(k)})_{j_p^k\phi}(X_{j_p^k\phi})) &= \pi_{k,l*}(v_{j_p^k\phi}(X_{j_p^k\phi}) \circledvee_{j_p^k\phi} \omega) \\
&= (\pi_{k-1,l-1*}(v_{j_p^k\phi}(X_{j_p^k\phi}))) \circledvee_{j_p^l\phi} \omega \\
&= (v_{j_p^l\phi}(Y_{j_p^l\phi})) \circledvee_{j_p^l\phi} \omega \\
&= (S_\omega^{(l)})_{j_p^l\phi}(Y_{j_p^l\phi}).
\end{aligned}$$

If now $\xi \in T_{j_p^k\phi}J^k\pi$ then

$$\begin{aligned}
(S_\omega^{(k)} \lrcorner \pi_{k,l}^*\sigma)_{j_p^k\phi}(\xi) &= \pi_{k,l}^*(\sigma_{j_p^l\phi})((S_\omega^{(k)})_{j_p^k\phi}(\xi)) \\
&= \sigma_{j_p^l\phi}\pi_{k,l*}((S_\omega^{(k)})_{j_p^k\phi}(\xi)) \\
&= \sigma_{j_p^l\phi}((S_\omega^{(l)})_{j_p^l\phi}(\pi_{k,l*}\xi)) \\
&= (S_\omega^{(l)} \lrcorner \sigma)_{j_p^l\phi}(\pi_{k,l*}\xi) \\
&= (\pi_{k,l}^*(S_\omega^{(l)} \lrcorner \sigma))_{j_p^k\phi}(\xi).
\end{aligned}$$

∎

With several closed 1-forms on M, one may consider combinations of S operators. As with vertical lifts, the dependence on the 1-forms is symmetric so that all the vertical endomorphisms commute. In this Lemma, as on some other occasions when referring to vertical endomorphisms on a fixed jet manifold $J^k\pi$, I shall omit the superscript k and refer simply to S_ω.

Lemma 5.2.8 *If $\omega^1, \omega^2 \in \bigwedge^1 M$ with $d\omega^1 = d\omega^2 = 0$ then*

$$S_{\omega^1} \lrcorner S_{\omega^2} = S_{\omega^2} \lrcorner S_{\omega^1}.$$

Proof If $\xi \in T_{j^k_p\phi}(J^k\pi)$,

$$\begin{aligned}(S_{\omega^1} \lrcorner S_{\omega^2})_{j^k_p\phi} &= (S_{\omega^2})_{j^k_p\phi}((S_{\omega^1})_{j^k_p\phi}(\xi)) \\ &= v_{j^k_p\phi}(v_{j^k_p\phi}(\xi) \oslash_{j^k_p\phi} \omega^1) \oslash_{j^k_p\phi} \omega^2.\end{aligned}$$

Now $v_{j^k_p\phi}(\xi) \oslash_{j^k_p\phi} \omega^1 \in V_{j^k_p\phi}\pi_{k,0} \subset V_{j^k_p\phi}\pi_k$, so that

$$\begin{aligned}v_{j^k_p\phi}(v_{j^k_p\phi}(\xi) \oslash_{j^k_p\phi} \omega^1) &= \pi_{k,k-1*}(v_{j^k_p\phi}(\xi) \oslash_{j^k_p\phi} \omega^1) \\ &= \pi_{k-1,k-2*}v_{j^k_p\phi}(\xi) \oslash_{j^{k-1}_p\phi} \omega^1\end{aligned}$$

by Lemma 5.2.3. Therefore

$$\begin{aligned}(S_{\omega^1} \lrcorner S_{\omega^2})_{j^k_p\phi}(\xi) &= \pi_{k-1,k-2*}v_{j^k_p\phi}(\xi) \oslash_{j^{k-1}_p\phi} \omega^1 \oslash_{j^k_p\phi} \omega^2 \\ &= \pi_{k-1,k-2*}v_{j^k_p\phi}(\xi) \oslash_{j^{k-1}_p\phi} \omega^2 \oslash_{j^k_p\phi} \omega^1 \\ &= (S_{\omega^2} \lrcorner S_{\omega^1})_{j^k_p\phi}(\xi).\end{aligned}$$

∎

As with vertical lifts, I shall use multi-index notation where a family of m closed 1-forms is available: for a multi-index I, S_{ω^I} is defined recursively by

$$S_{\omega^{I+1_i}} = S_{\omega^I} \lrcorner S_{\omega^i}.$$

Some examples:

- If π is the trivial bundle $(\mathbf{R} \times F, pr_1, \mathbf{R})$ and ω is the volume form dt on $\mathbf{R}$ then the derivatives of the coefficients of dt vanish, and in coordinates

$$S^{(k)}_{dt} = \sum_{r=0}^{k-1}(r+1)(dq^\alpha_{(r)} - q^\alpha_{(r+1)}dt) \otimes \frac{\partial}{\partial q^\alpha_{(r+1)}}$$

on $J^k\pi$. If $k = 1$ then

$$S_{dt}^{(1)} = (dq^\alpha - \dot{q}^\alpha dt) \otimes \frac{\partial}{\partial \dot{q}^\alpha}$$

on $J^1\pi = \mathbf{R} \times TF$, which is the contact equivalent of the original S tensor on TF.

- On the first jet manifold of any bundle (E, π, M) the vertical lift $\xi \circledvee_{j_p^1\phi} \omega$ and the vertical endomorphism $S_\omega^{(1)}$ do not depend on the derivatives of the coefficients of ω: in coordinates

$$\begin{aligned} \xi \circledvee_{j_p^1\phi} \omega &= \xi^\alpha \omega_i(p) \left.\frac{\partial}{\partial u_{1_i}^\alpha}\right|_{j_p^1\phi} \\ S_\omega^{(1)} &= \omega_i (du^\alpha - u_{1_j}^\alpha dx^j) \otimes \frac{\partial}{\partial u_{1_i}^\alpha}. \end{aligned}$$

This second example leads to a slight generalisation of the vertical endomorphism defined on the first jet manifold, where ω is now a 1-form on $J^1\pi$ horizontal over M.

Definition 5.2.9 *If $\omega \in \bigwedge_0^1 \pi_1$ then the first vertical endomorphism defined by ω is the vector-valued 1-form $S_\omega^{(1)}$ given by*

$$(S_\omega^{(1)})_{j_p^1\phi}(\xi) = v_{j_p^1\phi}(\xi) \circledvee_{j_p^1\phi} \overline{\omega}$$

where $\overline{\omega} \in \bigwedge^1 M$ and $\pi_1^* \overline{\omega}_p = \omega_{j_p^1\phi}$.

This definition makes sense because different choices of $\overline{\omega}$ give the same cotangent vector $\overline{\omega}_p$ and therefore the same vertical lift.

Example:

- If π is the trivial bundle $(M \times \mathbf{R}, pr_1, M)$ then, using the coordinate notation introduced previously for this example,

$$S_\omega^{(1)} = \omega_i (dt - p_j dq^j) \otimes \frac{\partial}{\partial p_i}.$$

However there is a canonical 1-form $\omega \in \bigwedge_0^1 \pi_1$, namely the 1-form defined by

$$\omega_{j_p^1\phi} = pr_1(j_p^1\phi) = d\overline{\phi}_p \in T_p^* M$$

so that

$$\omega_i(j_p^1\phi) = \left.\frac{\partial \overline{\phi}}{\partial q^i}\right|_p = p_i(j_p^1\phi)$$

and therefore $\omega = p_i dq^i$; ω is the jet version of the canonical 1-form on T^*M. The corresponding canonical vertical endomorphism is then given in coordinates by

$$S_\omega^{(1)} = p_i(dt - p_j dq^j) \otimes \frac{\partial}{\partial p_i}.$$

In particular,

$$\frac{\partial}{\partial t} \lrcorner\, S_\omega^{(1)} = p_i \frac{\partial}{\partial p_i} = \Delta,$$

the canonical dilation field on the vector bundle $\pi_{1,0}$.

5.3 Some Properties of Vertical Endomorphisms

I shall start this section by demonstrating that $S_\omega^{(k)}$ has certain properties which correspond to those of the original S tensor on TM or T^kM. The first property is a straightforward one, that if a vector field on $J^k\pi$ is vertical over $J^{l-1}\pi$ then contraction with $S_\omega^{(k)}$ gives a vector field which is vertical over $J^l\pi$. Loosely speaking, all the coordinates move up a level in the direction indicated by ω, and the coordinates with maximal length multi-indexes fall off the end.

Lemma 5.3.1 *If $\omega \in \bigwedge^1 M$ with $d\omega = 0$, and if $\xi \in V_{j_p^{k-1}\phi}\pi_{k-1,l-1}$ (where $1 \le l \le k$) then $\xi \circledvee_{j_p^k\phi} \omega \in V_{j_p^k\phi}\pi_{k,l}$.*

Proof Straightforward in coordinates. If

$$\xi = \sum_{|K|=0}^{k-1} \xi_K^\alpha \left.\frac{\partial}{\partial u_K^\alpha}\right|_{j_p^{k-1}\phi}$$

then $\xi_K^\alpha = 0$ whenever $|K| \le l-1$. But in the coordinate expression for $\xi \circledvee_{j_p^k\phi} \omega$ the coefficient of $\left.\frac{\partial}{\partial u_{J+K+1_i}^\alpha}\right|_{j_p^k\phi}$ always has a factor ξ_K^α, and so this term vanishes whenever $|K| \le l-1$ and consequently whenever $|J+K+1_i| \le l$. ∎

Corollary 5.3.2 *If $X \in \mathcal{V}(\pi_{k-1,l-1})$ then $X \circledvee \omega \in \mathcal{V}(\pi_{k,l})$.*

∎

Corollary 5.3.3 *If $X \in \mathcal{V}(\pi_{k,l-1})$ (where $1 \le l \le k$) then $X \lrcorner\, S_\omega^{(k)} \in \mathcal{V}(\pi_{k,l})$.*

∎

A final corollary to this result is that the $(k+1)$-th power of $S_\omega^{(k)}$ is identically zero, just as on T^kM. On jet bundles the result is in fact more general, because different 1-forms ω^i may be chosen for each iteration.

Corollary 5.3.4 *If $\omega^i \in \bigwedge^1 M$ with $d\omega^i = 0$ (where $1 \le i \le m$) and if I is a multi-index with $|I| > k$ then $S_{\omega^I}^{(k)} = 0$.*

∎

The statement about the rank of the vertical endomorphism on T^kM is not so easy to generalise to jet bundles, because there is the possibility that the 1-form ω may vanish at cerain points of M (or even, trivially, all over M). Furthermore the non-vanishing of the powers of the vertical endomorphism does not follow directly from the rank by a counting argument: for example if $m = 3$, $n = 1$ then $\dim J^2\pi = 13$ and the rank of $S_\omega^{(2)}$ will generically be 4, but in general $S_\omega^{(2)} \lrcorner\, S_\omega^{(2)} \neq 0$. I shall therefore consider these questions separately.

Lemma 5.3.5 *If $\omega \in \bigwedge^1 M$ with $d\omega = 0$, and if $p \in M$ with $\omega_p \neq 0$ then* $\operatorname{rank}(S_\omega^{(k)})_{j_p^k\phi} = n({}^{m+k-1}C_{k-1})$.

Proof Choose a coordinate system (x^i) in a neighbourhood of p such that $\omega = dx^1$; this is possible because $d\omega_p = 0$ and $\omega_p \neq 0$. The coordinate expression of $S_\omega^{(k)}$ then becomes

$$S_\omega^{(k)} = \sum_{|K|=0}^{k-1} (K(1) + 1)(du_K^\alpha - u_{K+1_j}^\alpha dx^j) \otimes \frac{\partial}{\partial u_{K+1_1}^\alpha}$$

so that the image space of $S_\omega^{(k)}$ at each point $j_p^k\phi$ in the fibre above p is spanned by $\left.\frac{\partial}{\partial u_{K+1_1}^\alpha}\right|_{j_p^k\phi}$ for $0 \le |K| \le k-1$. ∎

I shall now show that, in general, the k-th power of $S_{\omega^i}^{(k)}$ does not vanish.

Lemma 5.3.6 *If $\omega \in \bigwedge^1 M$ with $d\omega = 0$ and if $\omega_p \neq 0$ for some $p \in M$ then there is a vector field $X \in \mathcal{V}(\pi_{k-1,k-2})$ such that*

$$X \circledvee \omega \neq 0.$$

Proof Follows immediately from the coordinate representation of the vertical lift. ∎

Corollary 5.3.7 *If $\omega^i \in \bigwedge^1 M$ with $d\omega^i = 0$ (where $1 \leq i \leq m$) and if there is some $p \in M$ where each $\omega^i \neq 0$ then there is a vector field $X \in \mathcal{V}(\pi)$ such that, for any multi-index I,*

$$X \otimes \omega^I \neq 0.$$

Proof A straightforward induction argument using Corollary 5.3.2. ∎

Corollary 5.3.8 *If $\omega^i \in \bigwedge^1 M$ with $d\omega^i = 0$ (where $1 \leq i \leq m$) and if there is some $p \in M$ where each $\omega^i_p \neq 0$ then $S^{(k)}_{\omega^I} \neq 0$ for $|I| \leq k$.*

Proof I shall show that $S^{(k)}_{\omega^I} \neq 0$ whenever $|I| = k$, from which the result follows. So let $X \in \mathcal{V}(\pi)$ satisfy $X_{\phi(p)} \neq 0$ for some $\phi \in \Gamma_{loc}(\pi)$, and let $X^l \in \mathcal{V}(\pi_l)$ be the prolongation of X to $J^l\pi$ $(0 \leq l \leq k)$. Then $X^k \lrcorner S^{(k)}_{\omega^I} = X \otimes \omega^I$: for $X^k \lrcorner S^{(k)}_{\omega^i} = X^{k-1} \otimes \omega^i$, and if $X^k \lrcorner S^{(k)}_{\omega^J} = X^{k-|J|} \otimes \omega^J$ for $|J| < k$ then

$$\begin{aligned}
(X^k \lrcorner S^{(k)}_{\omega^{J+1_i}})_{j^k_p\phi} &= (X^k \lrcorner S^{(k)}_{\omega^J} \lrcorner S^{(k)}_{\omega^i})_{j^k_p\phi} \\
&= ((X^{k-|J|} \otimes \omega^J) \lrcorner S^{(k)}_{\omega^i})_{j^k_p\phi} \\
&= \pi_{k,k-1*}(X^{k-|J|} \otimes \omega^J)_{j^k_p\phi} \otimes_{j^k_p\phi} \omega^i \\
&= (X^{k-|J|-1} \otimes \omega^J)_{j^{k-1}_p\phi} \otimes_{j^k_p\phi} \omega^i \\
&= (X^{k-|J|-1} \otimes \omega^{J+1_i})_{j^k_p\phi}.
\end{aligned}$$

The result now follows from Corollary 5.3.7. ∎

The next set of properties concerns symmetries of $S^{(k)}_\omega$. I shall show that every prolongation of a bundle isomorphism is a symmetry of $S^{(k)}_\omega$. In fact these are not the only symmetries of $S^{(k)}_\omega$; there are other symmetries which are bundle isomorphisms of $\pi_{k,0}$ but which need not project onto M, just as there are diffeomorphisms of $J^k\pi$ which preserve the graphs of jets of sections but which are not prolongations. I shall not consider these more general symmetries here. I will, however, show that every infinitesimal symmetry of $S^{(k)}_\omega$ is a vector field on $J^k\pi$ which is the prolongation of a (not necessarily projectable) vector field on E.

Lemma 5.3.9 *If $(f, \overline{f})$ is a bundle isomorphism from π to itself, then*

$$f^k_*(\xi \otimes_{j^k_p\phi} \omega) = (f^{k-1}_*\xi) \otimes_{f^k(j^k_p\phi)} \omega.$$

Proof In the notation of Theorem 5.2.2,

$$\begin{aligned} f_*^{k-1}\xi &= f_*^{k-1}[t \longmapsto j_p^{k-1}\gamma_t] \\ &= [t \longmapsto f^{k-1}(j_p^{k-1}\gamma_t)] \\ &= [t \longmapsto j_{\overline{f}(p)}^{k-1}(f \circ \gamma_t \circ \overline{f}^{-1})] \end{aligned}$$

so that

$$\begin{aligned} (f_*^{k-1}\xi) \otimes_{f^k(j_p^k\phi)} \omega &= [t \longmapsto j_{\overline{f}(p)}^k(f \circ \chi_t \circ \overline{f}^{-1})] \\ &= [t \longmapsto f^k(j_p^k\chi_t)] \\ &= f_*^k[t \longmapsto j_p^k\chi_t] \\ &= f_*^k(\xi \otimes_{j_p^k\phi} \omega). \end{aligned}$$

∎

Corollary 5.3.10 *If $j_p^k\phi \in J^k\pi$ then $f_*^k \circ (S_\omega^{(k)})_{j_p^k\phi} = (S_\omega^{(k)})_{f^k(j_p^k\phi)} \circ f_*^k$.*

Proof If $\xi \in T_{j_p^k\phi}(J^k\pi)$ then

$$\begin{aligned} (f_*^k \circ (S_\omega^{(k)})_{j_p^k\phi})(\xi) &= f_*^k(v_{j_p^k\phi}(\xi) \otimes_{j_p^k\phi} \omega) \\ &= (f_*^{k-1}(v_{j_p^k\phi}(\xi)) \otimes_{j_p^k\phi} \omega \\ &= (v_{f^k(j_p^k\phi)}(f_*^k\xi)) \otimes_{j_p^k\phi} \omega \\ &= (S_\omega^{(k)})_{f^k(j_p^k\phi)}(f_*^k\xi). \end{aligned}$$

∎

Proposition 5.3.11 *If $X \in \mathcal{X}(E)$ is projectable onto M then $\mathcal{L}_{X^k}S_\omega^{(k)} = 0$; conversely if $Y \in \mathcal{X}(J^k\pi)$, $\omega \in \bigwedge^1 M$ does not vanish and $\mathcal{L}_Y S_\omega^{(k)} = 0$ then there is a vector field $X \in \mathcal{X}(E)$ such that $Y = X^k$.*

Proof By Proposition 3.2.17, if ψ_t is the flow of X then ψ_t^k is the flow of X^k. It follows immediately from Corollary 5.3.10 and Proposition 4.1.9 that $\mathcal{L}_{X^k}S_\omega^{(k)} = 0$.

Conversely, suppose that $\mathcal{L}_Y S_\omega^{(k)} = 0$. Then for any $\sigma \in \bigwedge^1 J^k\pi$, $d_Y(S_\omega^{(k)} \lrcorner\, \sigma) = S_\omega^{(k)} \lrcorner\, d_Y\sigma$; if also ω does not vanish then every contact form on $J^k\pi$ may be written as $S_\omega^{(k)} \lrcorner\, \sigma$ for a suitable choice of σ. Since $S_\omega^{(k)} \lrcorner\, d_Y\sigma$ is automatically a contact form, d_Y must preserve the module of contact forms on $J^k\pi$ and hence (by Proposition 3.2.11) Y is the prolongation of a vector field on E. ∎

In fact the Lie derivative $\mathcal{L}_{X^k} S_\omega^{(k)}$ will vanish even if X is not projectable onto M; however, the proof is more complicated and I shall omit it.

Next I shall consider the Nijenhuis tensor of $S_\omega^{(k)}$. Unfortunately, a simple example shows that the ordinary Nijenhuis tensor of $S_\omega^{(k)}$ does *not* vanish.

- On the trivial bundle $(\mathbf{R} \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates (x, u), $S_{dx}^{(1)}$ has coordinate representation

$$S_{dx}^{(1)} = (du - u_x dx) \otimes \frac{\partial}{\partial u_x}$$

  and so

$$\begin{aligned} \tfrac{1}{2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\right) \lrcorner N_{S_{dx}^{(1)}} &= \left[-u_x \frac{\partial}{\partial u_x}, \frac{\partial}{\partial u_x}\right] \\ &= \frac{\partial}{\partial u_x} \neq 0. \end{aligned}$$

The problem is that these vertical endomorphisms respect the contact structure of $J^k\pi$, which is ignored by the ordinary Nijenhuis tensor. The correct way of generalising the result from tangent bundle geometry is to bend the Nijenhuis tensor so that it, too, respects that structure. I shall therefore use the vertical bracket $[S_\omega^{(k)}, S_\omega^{(k)}]^v$ which I described at the end of Section 2.4. In fact, I shall prove a slightly more general result.

Proposition 5.3.12 *If $\omega^1, \omega^2 \in \bigwedge^1 M$ with $d\omega^1 = d\omega^2 = 0$ then*

$$[S_{\omega^1}^{(k)}, S_{\omega^2}^{(k)}]^v = 0.$$

Proof The derivation of type i_* corresponding to this vertical bracket is

$$\begin{aligned} & i_{S_{\omega^1}^{(k+1)}} \circ d_v \circ i_{S_{\omega^2}^{(k)}} + i_{S_{\omega^2}^{(k+1)}} \circ d_v \circ i_{S_{\omega^1}^{(k)}} \\ - \ & d_v \circ i_{S_{\omega^2}^{(k)}} \circ i_{S_{\omega^1}^{(k)}} - d_v \circ i_{S_{\omega^1}^{(k)}} \circ i_{S_{\omega^2}^{(k)}}. \end{aligned}$$

I claim that this derivation vanishes when acting on the coordinate 1-forms dx^i, du_I^α. For the image of any of these 1-forms under any of $i_{S_{\omega^1}^{(k)}}$, $i_{S_{\omega^2}^{(k)}}$ or

$$i_{S_{\omega^1}^{(k)}} \circ i_{S_{\omega^2}^{(k)}} = i_{S_{\omega^2}^{(k)}} \circ i_{S_{\omega^1}^{(k)}}$$

is the sum of contact forms $du_J^\alpha - u_{J+1_j}^\alpha dx^j$ multiplied by functions lifted from M (namely the derivatives of the coefficients of ω^1, ω^2); but

$$d_v(du_J^\alpha - u_{J+1_j}^\alpha dx^j) = 0$$

and if $f \in \pi_k^* C^\infty(M)$ then $d_v f = 0$. ∎

Finally in this section I shall mention how the endomorphisms $S_\omega^{(k)}$ may be used to show that the horizontal differential d_h is locally exact. A first remark is that the derivation of type h_* corresponding to $S_\omega^{(k)}$ is actually a derivation of type i_*: for if $f \in C^\infty(J^k\pi)$ then $i_{S_\omega^{(k)}} f = 0$ by definition, and $i_{S_\omega^{(k+1)}} d_h f = 0$ because $d_h f$ is horizontal over M. In fact, for any $\sigma \in \bigwedge^1 J^k\pi$, the relationship

$$i_{S_\omega^{(k+1)}} d_h \sigma - d_h i_{S_\omega^{(k)}} \sigma = \pi_{k+1}^* \omega \wedge i_v \sigma$$

may be obtained from a fairly lengthy calculation in local coordinates, and this suggests the possibility of attempting to construct a suitable homotopy operator for d_h using $S_\omega^{(k)}$. In [67], Tulczyjew defined certain coordinate-dependent operators acting on the differential forms on $J^\infty\pi$ called (in his notation) θ_m, where m was a multi-index; my notation would be to write such an operator as θ_I. Continuing in my notation, his definition was

$$\theta_0 = i_v$$

and for $|I| > 0$,

$$\begin{aligned} \theta_I(dx^i) &= 0, \\ \theta_I(du^\alpha - u^\alpha_{1_j} dx^j) &= 0, \\ \theta_I \circ d_{X_i} - d_{X_i} \circ \theta_I &= \begin{cases} \theta_{I-1_i} & \text{if } I(i) \geq 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where X_i is the coordinate total derivative $\frac{d}{dx^i}$. A property of these operators is that

$$\theta_I(du^\alpha_{I+J} - u^\alpha_{I+J+1_j} dx^j) = \frac{(I+J)!}{I!\,J!}(du^\alpha_J - u^\alpha_{J+1_j} dx^j)$$

from which it may be seen that θ_{1_i} is just a local version of the generalised vertical endomorphism $S_\omega^{(\infty)}$ where locally $\omega = dx^i$. The full construction of a homotopy operator for d_h using the operators θ_I is described in detail in [67].

5.4 The Vertical Vector-valued m-form on $J^1\pi$

In the final section of this chapter I shall show how the system of vertical endomorphisms defined in Section 5.2, with each vertical endomorphism depending on a closed 1-form on M, can on the first jet manifold $J^1\pi$ be replaced by a canonical vector-valued m-form which depends only on a choice of volume form Ω on the (assumed orientable) manifold M. In the course of this construction I have to turn the operator "inside out" to display a different aspect of what is essentially a single geometric object. It turns out that when this procedure is applied to the contact version of the original S tensor, $S^{(1)}_{dt}$ on $J^1\pi \cong \mathbf{R} \times TF$, one simply obtains $S^{(1)}_{dt}$ again. One single entity therefore plays a dual rôle, and it is this feature of the construction which explains why some of the properties of the S tensor as used in classical mechanics do not generalise to the corresponding operator in higher-order mechanics.

First, however, I shall show why there are problems when using this procedure on the jet manifolds $J^k\pi$ with $k \geq 2$. The difficulty arises in the nature of the single object which can be defined pointwise. On $J^1\pi$ this part of the construction is straighforward, for as I have already remarked the vertical endomorphism $(S^{(1)}_\omega)_{j^1_p\phi}$ only depends on the value of ω at the point p rather than in a neighbourhood of p. Consequently it is possible to define a single object $S^{(1)}$ which one might call a 2-contravariant, 1-covariant tensor field along π_1 in the spirit of the definition of vector-valued forms along bundle maps.

Definition 5.4.1 *The operator $S^{(1)}$ is defined to be the section of the bundle $T^*(J^1\pi) \otimes T(J^1\pi) \otimes \pi_1^* TM \longrightarrow J^1\pi$ specified by its pointwise action on two cotangent vectors on $J^1\pi$, one of which has been lifted from M:*

$$(S^{(1)})_{j^1_p\phi}(\eta, \mu) = (\check{S}^{(1)}_\omega)_{j^1_p\phi}(\eta)$$

where $\omega \in \bigwedge^1 M$ satisfies $\omega_p = \mu$ and $\check{S}^{(1)}_\omega$ denotes the adjoint action of $S^{(1)}_\omega$.

In coordinates,

$$S^{(1)} = (du^\alpha - u^\alpha_{1_j} dx^j) \otimes \frac{\partial}{\partial u^\alpha_{1_i}} \otimes \frac{\partial}{\partial x^i}$$

where the factor $\frac{\partial}{\partial x^i}$ is supposed to be a vector field along π_1. However, on higher jet manifolds one must take the following route. Given the closed 1-form $\omega \in \bigwedge^1 M$ and a point $p \in M$, let $f \in C^\infty(M)$ satisfy $df = \omega$ in some neighbourhood of p, and let ω^k be the section of the vector bundle of higher-order differentials $\tau^{*k}_M : T^{*k}M \longrightarrow M$ defined by

$$\omega^k_p = (d^k f)_p$$

where if $df = df'$ then also $d^k f = d^k f'$. Sections of the dual vector bundle $\tau^k_M : T^k M \longrightarrow M$ are linear differential operators on M, of order $\leq k$ ([69], p.20–22), so that in coordinates a section of τ^k_M would be of the form

$$X = \sum_{|I|=0}^{k} X^I \frac{\partial^{|I|}}{\partial x^I}.$$

(There is an unfortunate notational conflict here, in that the expression T^kM is used both for the total space of this vector bundle, containing linear differential operators at points of M, and for the total space of the higher-order tangent bundle—which is *not* a vector bundle—containing equivalence classes of curves with k-th order contact at the origin. I shall never mention both spaces in the same paragraph, and it should be clear from the context which one I mean.)

Definition 5.4.2 *The operator $S^{(k)}$ is defined to be the section of the bundle $T^*(J^k\pi) \otimes T(J^k\pi) \otimes \pi_k^* T^kM \longrightarrow J^k\pi$ specified by its pointwise action on a cotangent vector on $J^k\pi$ and a higher-order cotangent vector lifted from M:*

$$(S^{(k)})_{j_p^k\phi}(\eta, \mu) = (\check{S}^k_{df})_{j_p^k\phi}(\eta)$$

where $f \in C^\infty(J^k\pi)$ satisfies $\mu = d^k f_p$.

In coordinates,

$$S^{(k)} = \sum_{|J+K|=0}^{k-1} \frac{(J+K+1_i)!}{(J+1_i)!\,K!}(du^\alpha_K - u^\alpha_{K+1_j}dx^j) \otimes \frac{\partial}{\partial u^\alpha_{J+K+1_i}} \otimes \frac{\partial^{|J|+1}}{\partial x^{J+1_i}}.$$

So far, there is no real difference between the cases $k = 1$ and $k \geq 2$, apart from complexity (I do not wish to include the case $k = \infty$ in this part of my discussion as this would involve questions of convergence which would lead me too far afield). Equally, the next step applies generally to $1 \leq k < \infty$.

Definition 5.4.3 *Given $\sigma \in \bigwedge^1 J^k\pi$, the operator $S\sigma$ is defined by*

$$S\sigma = C(S^{(k)} \otimes \sigma)$$

where C is the operation of contracting the $T(J^k\pi)$ component of $S^{(k)}$ with σ.

Of course this, too, is a pointwise operation: $S\sigma$ is a section of the bundle $T^*(J^k\pi) \otimes \pi_k^* T^kM \longrightarrow J^k\pi$, and $(S\sigma)_{j_p^k\phi}$ depends only on the cotangent vector $\sigma_{j_p^k\phi}$ rather than on its extension as a 1-form. In coordinates, if $\sigma = \sigma_i dx^i + \sum_{|I|=0}^k \sigma^I_\alpha du^\alpha_I$ then

$$S\sigma = \sum_{|J+K|=0}^{k-1} \frac{(J+K+1_i)!}{(J+1_i)!\,K!}\sigma_\alpha^{J+K+1_i}(du^\alpha_K - u^\alpha_{K+1_j}dx^j) \otimes \frac{\partial^{|J|+1}}{\partial x^{J+1_i}}.$$

However, from this stage onwards the cases $k = 1$, $k \geq 2$ will be considered separately. When $k = 1$ the object $S\sigma$ is a vector-valued 1-form along π_1:

$$S\sigma = \sigma_\alpha^{1_i}(du^\alpha - u^\alpha_{1_j}dx^j) \otimes \frac{\partial}{\partial x^i}$$

and so may be used to define a derivation of type i_* along π_1. But when $k \geq 2$, $S\sigma$ represents a higher-order differential operator and so cannot be used in this way. For the final definition in this section I shall therefore restrict attention to the case $k = 1$, and defer further consideration of the cases $k \geq 2$ until the next Chapter.

Definition 5.4.4 *The canonical vector-valued m-form on $J^1\pi$ is denoted S_Ω and defined by its action on the 1-forms on $J^1\pi$:*

$$S_\Omega \lrcorner\, \sigma = i_{S\sigma}\Omega.$$

The coordinate representation of S_Ω is

$$S_\Omega = (du^\alpha - u^\alpha_{1_j} dx^j) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega \right) \otimes \frac{\partial}{\partial u^\alpha_{1_i}}.$$

In the particular case when $\dim M = 1$ with volume form dt then the vector-valued 1-form S_{dt} and the vertical endomorphism $S^{(1)}_{dt}$ are in fact identical, and it is reasonable to assert that an appropriate generalisation to the case when $\dim M > 1$ is the vector-valued m-form S_Ω rather than any particular vertical endomorphism $S^{(1)}_\omega$. Indeed, it is clear from the coordinate representation that S_Ω (and, also, $S\sigma$) are closely related to the operators θ and Θ defined in [32] and also described in [35], pages 202–210, where Θ is the Cartan form corresponding to a Lagrangian $L : J^1\pi \longrightarrow \mathbf{R}$. I shall describe the Cartan form in this and more general situations in Section 6.3; for the moment I shall simply observe that θ (which was constructed differently from $S\sigma$, using the affine structure of $\pi_{1,0}$) in fact equals $S(dL) + \frac{1}{m} L\Omega$ and that $\Theta = dL \lrcorner\, S_\Omega + L\Omega$.

Chapter 6

On the Calculus of Variations

The classical problem in the calculus of variations is to find those curves in the space of dependent variables $\mathbf{R}^n$ whose endpoints are fixed and which minimise (or at any rate provide extreme values of) the integral of a function L which depends on both the position of the curve and on its tangents. One particular example of this problem is to find the curves of shortest length on a surface embedded in $\mathbf{R}^3$, considering a portion of the surface sufficiently small that a coordinate system may be used. The curves then satisfy certain second-order differential equations, the "geodesic equations". A modern formulation of this classical problem may be given in terms of the tangent bundle of a manifold E.

In this Chapter I shall explain these ideas in more detail. Section 6.1 gives a brief summary of the relevant properties of these Lagrangian systems and their Hamiltonian counterparts, and in Section 6.2 I shall apply similar techniques to more general problems in the calculus of variations, where the unknown functions depend on several variables and where the Lagrangian may involve derivatives higher than the first. The natural setting for this later discussion is the theory of jet bundles, and the various manifestations of the generalised vertical endomorphism S in this extended context will be seen to be fundamental. In Section 6.3 I shall use it to construct a Cartan form, and in Section 6.4 I shall relate it to the second-order jet fields on π which were defined in Section 4.2.

A significant part of this chapter is original. Much of the content of Section 6.3 has been published in [63], and parts of the material in Section 6.4 are to appear in [65].

6.1 Lagrangian and Hamiltonian Systems

In this section I shall describe the relationship between variational problems for curves γ in a manifold E, Lagrangian systems on the tangent manifold

TE, and Hamiltonian systems on the cotangent manifold T^*E. A comprehensive account of this theory, expressed in the modern language of manifolds, may be found in [3]; my reason for repeating a small part of the theory in this dissertation is that I wish to place some of the constructions in a slightly different perspective.

I shall start with a function $L : TE \longrightarrow \mathbf{R}$. This is called a *Lagrangian* function, and I suppose given two fixed points $a_1, a_2 \in E$. The objective is to find curves $\gamma : [t_1, t_2] \longrightarrow E$ which satisfy $\gamma(t_1) = a_1$, $\gamma(t_2) = a_2$ and which provide extreme values of

$$\int_{t_1}^{t_2} L(\gamma'(t))dt$$

where $\gamma'(t)$ is the tangent vector in $T_{\gamma(t)}E$ which in the notation I have used previously would be written $[s \longmapsto \gamma(s+t)]$. It is well known that if a curve solves this problem then the *Euler-Lagrange equations* hold:

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} = 0$$

when evaluated along the curve. These equations form a family of n second-order ordinary differential equations which may be written slightly more explicitly as

$$\frac{\partial L}{\partial q^\alpha} - \dot{q}^\beta \frac{\partial^2 L}{\partial q^\beta \partial \dot{q}^\alpha} - \ddot{q}^\beta \frac{\partial^2 L}{\partial \dot{q}^\beta \partial \dot{q}^\alpha} = 0$$

and hence define a submanifold of the second-order tangent manifold T^2E. This formulation suggests the need for a suitable regularity condition, namely that the matrix of second derivatives $\frac{\partial^2 L}{\partial \dot{q}^\beta \partial \dot{q}^\alpha}$ (the *multiplier* matrix) is never singular; when this condition holds, the Lagrangian is called *regular* and the Euler-Lagrange equations may be solved for $\ddot{q}^\beta$. These quantities $\ddot{q}^\beta$ are the coefficients of a particular kind of vector field Γ_L on TE called a *second-order differential equation field.* In coordinates,

$$\Gamma_L = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \Gamma_L^\alpha \frac{\partial}{\partial \dot{q}^\alpha}.$$

There are several ways of characterising second-order differential equation fields. If Γ is a vector field on TE then it will be a second-order differential equation field if any one of the following three equivalent conditions hold:

1. Γ is a section of the bundle $\tau_{E*} : TTE \longrightarrow TE$;

2. $d_\Gamma S = \Delta$ where S is the almost tangent structure on TE and Δ is the dilation field; or

3. $\mathrm{Im}(\Gamma) \subset T^2E \subset TTE$, where I have identified T^2E with its image under the canonical inclusion $\iota_{1,1}$.

If γ is an extremal of L then γ' is an integral curve of the second-order differential equation field; the relationship between the vector field and the submanifold of T^2E is just a particular case of the relationship between a jet field and its associated differential equation described in Section 4.1.

An example:

- Suppose E is a pseudo-Riemannian manifold with metric g. Then the function $L : TE \longrightarrow \mathbf{R}$ defined by $L(a) = \frac{1}{2}g_{\tau_E(a)}(a, a)$ is a regular Lagrangian. In coordinates, if $g = g_{\alpha\beta}dq^\alpha \otimes dq^\beta$ and $a = a^\alpha \left.\frac{\partial}{\partial q^\alpha}\right|_{\tau_E(a)}$ then
$$L = \tfrac{1}{2}g_{\alpha\beta}(\tau_E(a))a^\alpha a^\beta = \left(\tfrac{1}{2}\tau_E^*(g_{\alpha\beta})\dot{q}^\alpha\dot{q}^\beta\right)(a)$$
and the Euler-Lagrange equations are (omitting the pull-back map τ_E^*)
$$\tfrac{1}{2}\left(\frac{\partial g_{\beta\gamma}}{\partial q^\alpha}\dot{q}^\gamma\dot{q}^\beta - \frac{\partial g_{\alpha\beta}}{\partial q^\gamma}\dot{q}^\beta\dot{q}^\gamma - \frac{\partial g_{\alpha\gamma}}{\partial q^\beta}\dot{q}^\beta\dot{q}^\gamma\right) - g_{\alpha\beta}\ddot{q}^\beta = 0$$
showing that regularity of the Lagrangian is a consequence of non-degeneracy of the metric. Solutions of these equations, when projected onto E, are the geodesics of the metric. The corresponding second-order differential equation field on TE is quadratic in the fibre coordinates $\dot{q}^\alpha$; a field with this quadratic property is called a *spray*, and this particular vector field is called the *geodesic spray* of the metric.

I shall return to Lagrangian systems shortly. First, however, I shall describe Hamiltonian systems, presenting an exposition in a slightly more general setting than T^*E with a view to subsequent applications.

Definition 6.1.1 *A* Hamiltonian structure *on a manifold M is a vector bundle map $\sharp : T^*M \longrightarrow TM$ over the identity on M which is symmetric and closed:*

1. $\sharp\omega_1 \lrcorner\, \omega_2 = -\sharp\omega_2 \lrcorner\, \omega_1$*;*
2. $\sum_{\sigma\in C_3} d_{\sharp\omega_{\sigma(1)}}(\sharp\omega_{\sigma(2)} \lrcorner\, \omega_{\sigma(3)}) - [\sharp\omega_{\sigma(1)}, \sharp\omega_{\sigma(2)}] \lrcorner\, \omega_{\sigma(3)} = 0$*, using C_3 to represent the group of cyclic permutations of* $\{1, 2, 3\}$*;*

where $\omega_i \in \bigwedge^1 M$ and I have also used the symbol $\sharp$ to indicate the induced mapping from 1-forms to vector fields.

An alternative way of describing a Hamiltonian structure is by using a bivector field Λ (that is, a skew-symmetric type (2,0) tensor field) and defining $\sharp$ by

$$\eta_1(\sharp\eta_2) = \Lambda_p(\eta_1, \eta_2)$$

where $p \in M$ and $\eta_1, \eta_2 \in T^*_p M$. Linearity and skew-symmetry of $\sharp$ are automatic; closure corresponds to the additional condition which must be imposed on Λ that the Schouten bracket $[\Lambda, \Lambda]$, which is a trivector field, vanishes.

Given a Hamiltonian structure on M, one may then define the Hamiltonian vector field corresponding to the function $f \in C^\infty(M)$; it is just $\sharp df$. Any vector field which may be written in this form is called *globally Hamiltonian*; if the vector field may be written in the form $\sharp\omega$ where ω is closed rather than exact then the vector field is called *locally Hamiltonian*. The *Poisson bracket* corresponding to the Hamiltonian structure $\sharp$ is then defined to be the mapping $C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ given by

$$\{f, g\} = -d_{\sharp df} g.$$

One may check that the Poisson bracket satisfies the conditions which make the vector space $C^\infty(M)$ an (infinite-dimensional) Lie algebra; in particular the Jacobi identity corresponds to the vanishing of the Schouten bracket of the bivector field Λ using the relationship $\{f, g\} = -\Lambda \lrcorner (df, dg)$. The Poisson bracket also satisfies the following derivation condition

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

which follows automatically from its definition in terms of the Lie derivative action of a vector field. Furthermore, the Poisson bracket of functions is related to the Lie bracket of vector fields by the formula

$$[\sharp df, \sharp dg] = -\sharp d\{f, g\}$$

so that the operator $\sharp \circ d : C^\infty(M) \longrightarrow \mathcal{X}(M)$ is a Lie algebra anti-homomorphism.

In fact this operation may be reversed. Starting with a Poisson bracket on M—that is, a Lie bracket on $C^\infty(M)$ satisfying the derivation property—one may define the Hamiltonian vector field Γ_f corresponding to the function f by its action as a derivation:

$$d_{\Gamma_f} g = -\{f, g\}.$$

One then constructs the Hamiltonian structure $\sharp$ by choosing, for each $\eta \in T_p M$, a function f such that $df_p = \eta$ and setting $\sharp\eta = (\Gamma_f)_p$ (this does not depend on the particular choice of f). Thus a Poisson bracket provides a third way of describing a Hamiltonian structure on M. The properties of the Poisson bracket may also be used to show that if Γ is a Hamiltonian vector field then $\mathcal{L}_\Gamma \Lambda = 0$; the proof follows by expanding $\mathcal{L}_\Gamma(\Lambda \lrcorner (df, dg))$, replacing the bivector field by the Poisson bracket and using the Jacobi identity. (Of course an equivalent calculation may be carried out directly with the Schouten bracket.)

An example:

- Suppose g is a Lie algebra. There is a natural Poisson bracket on the dual vector space g^*, which may be defined using the isomorphism $g^{**} \cong g$. If $f, g \in C^\infty(g^*)$ then define $\{f, g\}$ by

$$\{f, g\}(p) = [df_p, dg_p](p)$$

  for $p \in g^*$. Here, $df_p, dg_p \in T_p^* g^* \cong g^{**} \cong g$ so that the Lie bracket of df_p and dg_p makes sense; the result, an element of g, may then be regarded as a linear function on g^* and so evaluated at p. A similar construction may also be used to define a Poisson bracket if g is not finite-dimensional, so long as it is a reflexive topological Lie algebra; however in these circumstances there are derivations on the ring of functions which are not vector fields (that is, they cannot be represented as sections of the tangent bundle).

The *rank* of a Hamiltonian structure $\sharp$ at each point $p \in M$ is simply its rank as a linear mapping at that point. If the rank of $\sharp$ equals the dimension of M at each point then the Hamiltonian structure has the additional property of being *symplectic*. For the mapping $\sharp$ will then be invertible, and if its inverse is denoted by $\flat$ then one may define a non-degenerate 2-form ω by

$$\omega_p(\xi_1, \xi_2) = (\flat\xi_1)(\xi_2)$$

where $p \in M$ and $\xi_1, \xi_2 \in T_p M$. The skew-symmetry and non-degeneracy of ω follow from the corresponding properties of $\sharp$, and similarly the closure condition on $\sharp$ implies that $d\omega$ vanishes so that ω is indeed symplectic. Conversely, any symplectic form yields a Hamiltonian structure of maximal rank. Furthermore, if Γ is a Hamiltonian vector field on a symplectic manifold with symplectic form ω then $d_\Gamma \omega = 0$: for if $\Gamma = \sharp dH$ then $d_\Gamma \omega = -d(i_\Gamma \omega) = -d(dH) = 0$.

Some examples:

- The co-adjoint action of a Lie group G on the dual space g^* of its Lie algebra g determines a foliation of g^* into orbits of the action. The

Hamiltonian vector fields given by the natural Poisson bracket on g^* are tangent to the leaves of this foliation, and indeed the rank of the Hamiltonian structure at any point equals the dimension of the leaf containing that point. The Hamiltonian structure, when restricted to a leaf, is then of maximal rank and so determines a symplectic form on the leaf. Indeed, any manifold with a Hamiltonian structure has a symplectic foliation constructed in a similar way from integral manifolds of the system of Hamiltonian vector fields.

- Every cotangent manifold T^*E carries a natural symplectic structure ω given by the exterior derivative of the 1-form θ defined by

$$\theta_a(\xi) = a((\tau_E^*)_*(\xi))$$

where $a \in T^*E$, $\xi \in T_aT^*E$ and $\tau_E^* : T^*E \longrightarrow E$ is the canonical projection. In coordinates (q^α, p_α) on T^*E, $\theta = p_\alpha dq^\alpha$ and $\omega = dp_\alpha \wedge dq^\alpha$; indeed Darboux' Theorem asserts that on an arbitrary symplectic manifold one may always find local coordinates such that ω takes this form. Given a function $H : M \longrightarrow \mathbf{R}$, called a Hamiltonian function, the integral curves of the Hamiltonian vector field $\sharp dH$ are solutions of Hamilton's equations

$$\begin{aligned} \dot{q}^\alpha &= \frac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha &= -\frac{\partial H}{\partial q^\alpha}. \end{aligned}$$

The system is termed *completely integrable* if $\frac{1}{2}\dim M$ functions f^α may be found which are *independent* (that is, at each point $p \in M$, apart perhaps from a set of measure zero, the cotangent vectors df_p^α are linearly independent); which are *first integrals* (that is, $d_{\sharp dH} f^\alpha = 0$ so that the functions are constant along the integral curves of $\sharp dH$); and which are *in involution* (that is, $\{f^\alpha, f^\beta\} = 0$). Of course, H itself may be taken as one of the first integrals. Finally, I remark that the Hamiltonian flow depends, not on the function H as such, but on an equivalence class of such functions differing by what are often called *Casimir functions*; for a non-degenerate Hamiltonian structure the only such functions are constants. More precisely, if one considers the initial part of the de Rham complex

$$0 \longrightarrow \mathbf{R} \longrightarrow \bigwedge^0 M \longrightarrow \bigwedge^1 M \longrightarrow \dots$$

then two functions are equivalent in the non-degenerate case if their difference is in the image of the map $\mathbf{R} \longrightarrow \bigwedge^0 M$. This point of view is not just pedantry but will be a guide to the interpretation of Hamiltonian systems of evolution equations in Section 7.1.

- One may also consider the possibility of using two different Hamiltonian structures $\sharp, \widetilde{\sharp}$ on the same manifold M and enquiring whether a vector field Γ might be Hamiltonian with respect to both of these structures, so that $\Gamma = \sharp dH = \widetilde{\sharp} d\widetilde{H}$. This would be called a *bi-Hamiltonian system.* (Clearly the case $\widetilde{\sharp} = \lambda\sharp$, $\lambda \in \mathbf{R}$, is trivial, so I shall assume that this is not the case.) If one of the Hamiltonian structures, say $\sharp$, is non-singular with inverse $\flat$ then the composition $\widetilde{\sharp} \circ \flat$ is a vector bundle map $TM \longrightarrow TM$ over the identity on M and hence defines a vector-valued 1-form R. If the bivector fields corresponding to $\sharp, \widetilde{\sharp}$ are denoted $\Lambda, \widetilde{\Lambda}$ then from $\mathcal{L}_\Gamma \Lambda = \mathcal{L}_\Gamma \widetilde{\Lambda} = 0$ one easily obtains $\mathcal{L}_\Gamma R = 0$ and hence $\mathcal{L}_\Gamma R^n = 0$ for each $n \in \mathbf{N}$. Consequently the trace of each R^n is a first integral of Γ. On the other hand, if a Hamiltonian vector field Γ on a symplectic manifold is completely integrable then it is locally bi-Hamiltonian. See, for example, [4,8] for a further discussion of these points.

I shall now return to the Lagrangian description of the calculus of variations and show how it is related to a Hamiltonian structure. So suppose $L : TE \longrightarrow \mathbf{R}$ is a hyper-regular Lagrangian. One may construct a map called the *fibre derivative* of L, $\mathcal{F}L : TE \longrightarrow T^*E$, in the following way: for each $a \in E$,

$$\mathcal{F}L_a(\xi) = d(L|_{T_aE})_\xi \in T^*_\xi T_aE \cong T^*_aE$$

where $\xi \in T_aE$. This map allows the canonical forms θ and ω on T^*E to be pulled back to TE; in coordinates

$$(\mathcal{F}L)^*\theta = \frac{\partial L}{\partial \dot{q}^\alpha} dq^\alpha$$

and

$$(\mathcal{F}L)^*\omega = \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} dq^\beta \wedge dq^\alpha + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta} d\dot{q}^\beta \wedge dq^\alpha.$$

Writing ω_L for $(\mathcal{F}L)^*\omega$, one finds that the second-order differential equation field Γ_L is globally Hamiltonian with respect to the Hamiltonian structure corresponding to the symplectic form ω_L, and that the corresponding Hamiltonian function is the energy $d_\Delta L - L$, where Δ is the dilation field on TE given by its vector bundle structure over E. An alternative method of obtaining ω_L which does not involve explicit use of the cotangent bundle is by writing it as $dd_S L$, where S is the canonical almost tangent structure on TE [11]. One may also consider bi-Lagrangian systems; these, however, are more restrictive than bi-Hamiltonian systems as the two symplectic forms

$\omega_L, \omega_{\tilde{L}}$ must both be constructed in the prescribed way from the Lagrangians $L, \tilde{L}$.

A similar construction to the above may be performed when the Lagrangian is explicitly time dependent, and so is a map $L : \mathbf{R} \times TE \cong J^1\pi \longrightarrow \mathbf{R}$ where $\pi : \mathbf{R} \times E \longrightarrow \mathbf{R}$. The corresponding second-order differential equation field has the coordinate representation

$$\frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \Gamma^\alpha \frac{\partial}{\partial \dot{q}^\alpha}$$

where now

$$\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta} \Gamma^\beta = \frac{\partial L}{\partial q^\alpha} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\alpha} - \dot{q}^\beta \frac{\partial^2 L}{\partial q^\beta \partial \dot{q}^\alpha}.$$

Writing S now for the vertical vector-valued 1-form S_{dt} described in Section 5.2 one finds that, in coordinates,

$$\begin{aligned} d_S L + L\, dt &= \frac{\partial L}{\partial \dot{q}^\alpha}(dq^\alpha - \dot{q}^\alpha dt) + L\, dt \\ &= \frac{\partial L}{\partial \dot{q}^\alpha} dq^\alpha - H\, dt \end{aligned}$$

where, as before, $H = d_\Delta L - L$ is the energy of L. This 1-form, denoted Θ_L, is called the *Cartan form* of L. Its exterior derivative $d\Theta_L$ satisfies

$$\Gamma_L \lrcorner\, d\Theta_L = 0;$$

in other words, Γ_L is a characteristic vector field of the 2-form $d\Theta_L$. In subsequent sections of this chapter I shall show how a Cartan form may be constructed in more general situations and I shall describe a higher-dimensional analogue of the equation $\Gamma_L \lrcorner\, d\Theta_L = 0$.

6.2 Variational Principles and Lagrangian Field Theories

As before, I shall assume that the base manifold M of the bundle (E, π, M) is orientable with volume form Ω; I shall use the same symbol Ω to denote the pullback $\pi_k^* \Omega$ on the jet manifold $J^k\pi$.

Definition 6.2.1 *A* k-th order Lagrangian on π *is a function* $L \in C^\infty(J^k\pi)$. *The corresponding* Lagrangian density *is the m-form* $L\Omega \in \bigwedge_0^m \pi_k$.

Given a fixed Lagrangian L, each $\phi \in \Gamma_{loc}(\pi)$ determines a function $(j^k\phi)^* L : \mathrm{domain}(\phi) \longrightarrow \mathbf{R}$. I shall be interested in the integrals of such functions; the following development of the definition of ϕ as an extremal of L is similar to that in [43].

Definition 6.2.2 *If $\phi \in \Gamma_{loc}(\pi)$ and $X \in \mathcal{V}(\pi)$ with flow ψ_t, the* variation of ϕ induced by X *is the one-parameter family of local sections $\widetilde{\psi_t}(\phi) \in \Gamma_{loc}(\pi)$.*

For small t, the variation of ϕ is therefore a "nearby" section, a generalisation of the "nearby" curve used in the classical calculus of variations; indeed when $p \in \mathrm{domain}(\phi)$ the tangent vector $[t \longmapsto \widetilde{\psi_t}(\phi)(p)]$ just equals $(\tilde{X}\phi)_p$ where $\tilde{X}\phi$ is the vector field along ϕ defined in Section 1.2.

If C is a compact, simply-connected m-dimensional submanifold of M and $\phi \in \Gamma_{loc}(\pi)$ with $C \subset \mathrm{domain}(\phi)$ then a function $(-\epsilon, \epsilon) \longrightarrow \mathbf{R}$ for some $\epsilon > 0$ may be defined by

$$t \longmapsto \int_C (j^k(\psi_t \circ \phi))^* L\Omega.$$

The local section ϕ will be called an extremal of L if this function is stationary for every $C \subset \mathrm{domain}(\phi)$ and every X which vanishes on $\pi^{-1}(\partial C)$, where ∂C is the boundary of C.

Definition 6.2.3 *$\phi \in \Gamma_{loc}(\pi)$ is an* extremal of L *if*

$$\left.\frac{d}{dt}\right|_{t=0} \int_C (j^k(\psi_t \circ \phi))^* L\Omega = 0$$

whenever C is a compact, simply-connected m-dimensional submanifold of M, $C \subset \mathrm{domain}(\phi)$ $X \in \mathcal{V}(\pi)$, $X|_{\partial C} = 0$ and ψ_t is the flow of X.

Lemma 6.2.4 *ϕ is an extremal of L if, and only if,*

$$\int_C (j^k\phi)^* d_{X^k} L\Omega = 0.$$

Proof For each $p \in C$,

$$\begin{aligned}
(j^k\phi)^* d_{X^k} L(p) &= X^k_{j^k_p\phi} L \\
&= [t \longmapsto \psi^k_t(j^k_p\phi)](L) \\
&= [t \longmapsto j^k(\psi_t \circ \phi)(p)](L) \\
&= \left.\frac{d}{dt}\right|_{t=0} (j^k(\psi_t \circ \phi))^* L(p)
\end{aligned}$$

so that

$$\int_C (j^k\phi)^* d_{X^k} L\Omega = \left.\frac{d}{dt}\right|_{t=0} \int_C (j^k(\psi_t \circ \phi))^* L\Omega.$$

∎

As an example of the way in which this characterisation of extremals may be used to obtain the local Euler-Lagrange equations, I shall illustrate the technique for first-order Lagrangians.

Proposition 6.2.5 *Given $L \in C^\infty(J^1\pi)$, let C be a fixed compact simply-connected m-dimensional submanifold of M lying within a single coordinate patch. Suppose $\phi \in \Gamma_{loc}(\pi)$ and $C \subset$ domain(ϕ), with $\phi(C)$ again lying within a single coordinate patch. If ϕ is an extremal of L then ϕ satisfies the Euler-Lagrange equations*

$$(j^2\phi)^* \left(\frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i} \frac{\partial L}{\partial u^\alpha_{1_i}} \right) = 0 \qquad 1 \leq \alpha \leq n$$

at every interior point of C.

Proof Let the vector field $X \in \mathcal{V}(\pi)$ satisfy $X|_{\pi^{-1}(\partial C)} = 0$. If the coordinate representation of X is $X^\alpha \frac{\partial}{\partial u^\alpha}$ then $X^\alpha(a) = 0$ whenever $\pi(a) \in \partial C$. Consequently

$$\int_{\partial C} (j^1\phi)^* X^\alpha \frac{\partial L}{\partial u^\alpha_{1_i}} \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega \right) = 0.$$

I shall now apply Stokes' Theorem to this integral, together with Lemma 2.4.3.

$$\begin{aligned}
&\int_{\partial C} (j^1\phi)^* X^\alpha \frac{\partial L}{\partial u^\alpha_{1_i}} \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega \right) \\
= &\int_C d\left((j^1\phi)^* X^\alpha \frac{\partial L}{\partial u^\alpha_{1_i}} \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega \right) \right) \\
= &\int_C (j^2\phi)^* d_h \left(X^\alpha \frac{\partial L}{\partial u^\alpha_{1_i}} \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega \right) \right) \\
= &\int_C (j^2\phi)^* \left(\frac{\partial L}{\partial u^\alpha_{1_i}} d_h X^\alpha + X^\alpha d_h \left(\frac{\partial L}{\partial u^\alpha_{1_i}} \right) \right) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega \right) \\
= &\int_C (j^2\phi)^* \left(\frac{\partial L}{\partial u^\alpha_{1_i}} \frac{dX^\alpha}{dx^i} + X^\alpha \frac{d}{dx^i} \frac{\partial L}{\partial u^\alpha_{1_i}} \right) \Omega
\end{aligned}$$

where as usual I have omitted the various projection maps, so that (for example) the symbol X^α represents three functions on the manifolds E, $J^1\pi$ and $J^2\pi$.

Now from the characterisation of extremals,

$$\begin{aligned}
0 &= \int_C (j^1\pi)^* d_{X^1} L\Omega \\
&= \int_C (j^1\phi)^* \left(X^\alpha \frac{\partial L}{\partial u^\alpha} + \frac{dX^\alpha}{dx^i} \frac{\partial L}{\partial u^\alpha_{1_i}} \right) \Omega \\
&= \int_C (j^2\phi)^* \left(X^\alpha \left(\frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i} \frac{\partial L}{\partial u^\alpha_{1_i}} \right) \right) \Omega.
\end{aligned}$$

One then shows in the usual way, by taking a suitable variation field X, that if p is any interior point of C then the vanishing of this integral implies that

$$(j^2\phi)^*\left(\frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i}\frac{\partial L}{\partial u^\alpha_{1_i}}\right)(p) = 0.$$

∎

Some examples:

- Suppose π is the trivial bundle $(\mathbf{R}^2 \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates $(x, y; u)$ and that $L \in C^\infty(J^1\pi)$ is given by

  $$L = \tfrac{1}{2}u_x u_y - \cos u.$$

  The Euler-Lagrange equation corresponding to this Lagrangian is

  $$(j^2\phi)^*(\sin u - u_{xy}) = 0$$

  which, rewritten, is the sine-Gordon equation

  $$\frac{\partial^2\phi}{\partial x\,\partial y} = \sin\phi.$$

- More generally, suppose π is the trivial bundle $(M \times F, pr_1, M)$ where both M and F are Riemannian manifolds with metrics g, h respectively. Let $\hat{g}$ be the contravariant metric on M corresponding to g (so that $\hat{g}$ is a symmetric type (2,0) tensor field with the property that the map $g^\sharp : \bigwedge^1 M \longrightarrow \mathcal{X}(M)$ defined by $g^\sharp(\sigma_1) \lrcorner\, \sigma_2 = \hat{g}(\sigma_1, \sigma_2)$ is the inverse of the map $g^\flat : \mathcal{X}(M) \longrightarrow \bigwedge^1 M$ defined by $X_1 \lrcorner\, g^\flat(X_2) = g(X_1, X_2)$). For each $\phi \in \Gamma_{loc}(\pi)$, define the energy density of ϕ at $p \in \text{domain}(\phi)$ by

  $$e[\phi](p) = \tfrac{1}{2}\hat{g}_p(\phi^* pr_2^* h)_p.$$

  In coordinates, if $\hat{g} = g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ and $h = h_{\alpha\beta}du^\alpha \otimes du^\beta$ then

  $$e[\phi] = \tfrac{1}{2}g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}h_{\alpha\beta}$$

  and so e determines a function $e^1 \in C^\infty(J^1\pi)$ given in coordinates by

  $$e^1 = \tfrac{1}{2}g^{ij}h_{\alpha\beta}u^\alpha_{1_i}u^\beta_{1_j}.$$

  The energy of a section ϕ over a compact $C \subset M$ is

  $$\int_C e[\phi]\Omega = \int_C (j^1\phi)^* e^1\Omega.$$

Extremals of e^1 are the graphs of maps $M \longrightarrow F$ known as *harmonic maps*; the Euler-Lagrange equations for such maps may be calculated as

$$(j^2\phi)^*(g^{ij}(u^{\alpha}_{1_i+1_j} - {}^g\Gamma^k_{ij}u^{\alpha}_{1_k} + {}^h\Gamma^{\alpha}_{\beta\gamma}u^{\beta}_{1_i}u^{\gamma}_{1_j}))$$

where ${}^g\Gamma$, ${}^h\Gamma$ are the Christoffel symbols of the Levi-Cività connections of g and h. In the particular case when $\dim M = 1$ then $e^1 = \frac{1}{2}h_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta}$ and the Euler-Lagrange equations become

$$(j^2\phi)^*(\ddot{q}^{\alpha} + {}^h\Gamma^{\alpha}_{\beta\gamma}\dot{q}^{\beta}\dot{q}^{\gamma}) = 0$$

the geodesic equations in F. An extensive description of harmonic maps and their properties may be found in [18].

6.3 The Cartan Form

In Proposition 6.2.5 I demonstrated how the Euler-Lagrange equations for a first-order Lagrangian system could be constructed in a local coordinate patch. A major objective of the theory is to carry out a similar construction globally, and also to generalise the procedure to higher-order Lagrangians. To demonstrate how this may be done, I shall examine in more detail the operations carried out to the integrand in that Proposition.

Starting with the m-form $d_{X^k}L\Omega$ on $J^1\pi$, I effectively lifted this form to $J^2\pi$ and then subtracted from it $d_h\Theta^X_L$, where Θ^X_L denotes the $(m-1)$-form on $J^1\pi$ written in coordinates as

$$\frac{\partial L}{\partial u^{\alpha}_{1_i}}X^{\alpha}\left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega\right).$$

Since the variation field X vanished at points corresponding to the boundary of the region of integration, $d_h\Theta^X_L$ made no contribution to the integral; the purpose of the subtraction was to produce an integrand which did not involve derivatives of the coefficient functions X^{α}. If I write E^X_L for the m-form on $J^2\pi$ written in coordinates as

$$X^{\alpha}\left(\frac{\partial L}{\partial u^{\alpha}} - \frac{d}{dx^i}\frac{\partial L}{\partial u^{\alpha}_{1_i}}\right)\Omega$$

then my operation on the integrand has been to write

$$E^X_L = \pi^*_{2,1}d_{X^1}L\Omega - d_h\Theta^X_L$$

where for clarity I have reinstated the pull-back map $\pi^*_{2,1}$, and the various differential forms involved satisfy $X^X_L \in \bigwedge^m_0 \tilde{\pi}_2$, $d_{X^1}L\Omega \in \bigwedge^m_0 \tilde{\pi}_1$ and $\Theta^X_L \in$

$\bigwedge_0^{m-1} \tilde{\pi}_1$, where the tilde indicates the restriction of the jet projection to a suitably small portion of the appropriate jet manifold. This equation (or a closely related one) is commonly called the *equation of first variation*, and the key to preparing a global version of this construction is to note that each of the three m-forms involved may be regarded as the contraction of a suitable $(m+1)$-form with the second prolongation X^2 of the variation field:

$$\begin{aligned} E_L^X &= X^2 \lrcorner \, \delta L \\ \pi_{2,1}^* d_{X^1} L\Omega &= X^2 \lrcorner \, \pi_{2,1}^* dL \wedge \Omega \\ d_h \Theta_L^X &= -X^2 \lrcorner \, d_h \Theta_L \end{aligned}$$

where I shall describe δL and Θ_L shortly (the choice of sign for Θ_L is purely conventional). Since the variation field X was chosen arbitrarily, the equation of first variation may be written

$$\delta L = \pi_{2,1}^* dL \wedge \Omega + d_h \Theta_L$$

where this equation is now required to hold globally on $J^2\pi$.

Since the $(m+1)$-form $\delta L \in \bigwedge_1^{m+1} \pi_2$ must have the property that $X^2 \lrcorner \, \delta L$ does not involve the derivatives of the functions X^α, δL must be horizontal over E; in coordinates

$$\delta L = \left(\frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i} \frac{\partial L}{\partial u^\alpha_{1_i}} \right) du^\alpha \wedge \Omega \in \bigwedge\nolimits_0^{m+1} \pi_{2,0}.$$

I shall call δL the *Euler-Lagrange form* of L. The m-form $\Theta_L \in \bigwedge_1^m \pi_1$ must then be chosen so that δL has this property. There are many possible choices of Θ_L which give this result; however, as I shall show later (Proposition 6.3.7) there is only one such form which also has the same extremals as L in the sense that $(j^1\phi)^* L\Omega = (j^1\phi)^*\Theta_L$ for every $\phi \in \Gamma_{loc}(\pi)$. This unique m-form on $J^1\pi$ will be called the *Cartan form* of L; in coordinates it is

$$\Theta_L = \frac{\partial L}{\partial u^\alpha_{1_i}} (du^\alpha - u^\alpha_{1_j} dx^j) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \, \Omega \right) + L\Omega.$$

The global construction is carried out readily with the aid of the canonical vector-valued m-form on $J^1\pi$ described in Section 5.4.

Theorem 6.3.1 *If $L \in C^\infty(J^1\pi)$ then the Cartan form of L may be defined globally by*

$$\Theta_L = d_{S_\Omega} L + L\Omega.$$

Proof It is clear from the coordinate representation

$$S_\Omega \lrcorner\, \sigma = \sigma_\alpha^{1_i}(du^\alpha - u_{1_j}^\alpha dx^j) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega\right)$$

where $\sigma \in \bigwedge^1 J^1\pi$ that the following properties are satisfied:

1. $(S_\Omega \lrcorner\, \sigma)_{j_p^1\phi}$ depends only upon the germ of σ at $j_p^1\phi$;
2. $\pi_{2,1}^*(\sigma \wedge \Omega) + d_h(S_\Omega \lrcorner\, \sigma) \in \bigwedge_0^{m+1}\pi_{2,0} \cap \bigwedge_1^{m+1}\pi_2$; and
3. $(j^1\phi)^*(S_\Omega \lrcorner\, \sigma) = 0$ for every $\phi \in \Gamma_{loc}(\pi)$.

where these conditions have been selected so that they may be generalised later to higher-order systems. It is then immediate that Θ_L has the properties required of a Cartan form; the definition is global because S_Ω has been defined globally. ∎

Some examples:

- Suppose π is the trivial bundle $(\mathbf{R}^2 \times \mathbf{R}, pr_1, \mathbf{R}^2)$ with coordinates $(x, y; u)$ and that $L \in C^\infty(J^1\pi)$ is given as before by

  $$L = \tfrac{1}{2}u_x u_y - \cos u.$$

  Then the Cartan form of L is

  $$\Theta_L = \tfrac{1}{2}du \wedge (u_y dy - u_x dx) - (\tfrac{1}{2}u_x u_y + \cos u)dx \wedge dy.$$

- Now suppose that π is the trivial bundle $(M \times F, pr_1, M)$ where, as before, M and F are Riemannian manifolds and that $e^1 \in C^\infty(J^1\pi)$ is the energy density mapping. Then the Cartan form of e^1 is

  $$\Theta_e = g^{ij}h_{\alpha\beta}u_{1_j}^\beta(du^\alpha - \tfrac{1}{2}u_{1_k}^\alpha dx^k) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega\right).$$

I shall now carry out a similar construction for higher-order Lagrangians. By analogy with the first-order case, my objective will be to construct an equation of first variation in the form

$$\delta L = \pi_{2k,k}^* dL \wedge \Omega + d_h\Theta_L$$

where the Euler-Lagrange form δL will be an element of $\bigwedge_1^{m+1}\pi_{2k} \cap \bigwedge_0^{m+1}\pi_{2k,0}$; the reasons why the coefficient functions of δL are regarded as being defined on $J^{2k}\pi$ is that the construction involves "integration by parts" k times. The Cartan form Θ_L will then turn out to be an element of $\bigwedge_1^m \pi_{2k-1} \cap \bigwedge_0^m \pi_{2k-1,k-1}$ which also has the property that $(j^{2k-1}\phi)^*\Theta_L = (j^k\phi)^* L\Omega$ for every $\phi \in \Gamma_{loc}(\pi)$. However, the construction I have employed for first-order theories cannot be used directly in the higher-order case because the map $S\sigma$ introduced in Definition 5.4.3 becomes a differential operator and cannot be used directly to construct a map S_Ω. Instead I shall adapt the technique of Kuperschmidt [44] and use induction on the order k by taking advantage of the injection $\iota_{k-1,1} : J^k\pi \longrightarrow J^{k-1}\pi_1$ defined in Proposition 3.3.1.

I shall first prove two technical lemmas.

Lemma 6.3.2 *There is a canonical map (also denoted S_Ω) from $\bigwedge_1^{m+1} \pi_1$ to $\bigwedge_0^m \pi_{1,0} \cap \bigwedge_1^m \pi_1$ which satisfies $S_\Omega(\sigma \wedge \Omega) = S_\Omega \lrcorner \sigma$ for $\sigma \in \bigwedge^1 J^1\pi$, and $(j^1\phi)^*(S_\Omega(\theta)) = 0$ for $\theta \in \bigwedge_1^{m+1} \pi_1$ and $\phi \in \Gamma_{loc}(\pi)$.*

Proof Suppose $\theta \in \bigwedge_1^{m+1} \pi_1$. As in Proposition 1.3.12, define $\sigma \in \bigwedge^1 J^1\pi$ to be a representative of θ if $\theta = \sigma \wedge \Omega$. If σ_1, σ_2 are both representatives of θ then $(\sigma_1 - \sigma_2) \wedge \Omega = 0$ so that $\sigma_1 - \sigma_2 \in \bigwedge_0^1 \pi_1$ and hence $S_\Omega \lrcorner \sigma_1 = S_\Omega \lrcorner \sigma_2$. I may therefore define $S_\Omega(\theta)$ to equal $S_\Omega \lrcorner \sigma$ where σ is any representative of θ. ∎

Lemma 6.3.3 *The map $S_\Omega : \bigwedge_1^{m+1} \pi_1 \longrightarrow \bigwedge_0^m \pi_{1,0} \cap \bigwedge_1^m \pi_1$ can be lifted to a map (also called S_Ω) from $\bigwedge_0^{m+1} \pi_{s,1} \cap \bigwedge_1^{m+1} \pi_s$ to $\bigwedge_0^m \pi_{s,0} \cap \bigwedge_1^m \pi_s$, satisfying $(j^s\phi)^*(S_\Omega(\theta)) = 0$ for $\phi \in \Gamma_{loc}(\pi)$.*

Proof $\bigwedge_0^{m+1} \pi_{s,1} \cap \bigwedge_1^{m+1} \pi_s$ is the module generated by $\pi_{s,1}^*(\bigwedge_1^{m+1} \pi_1)$ over $C^\infty(J^s\pi)$, so define $S_\Omega(\pi_{s,1}^*(\theta))$ to equal $\pi_{s,1}^*(S_\Omega(\theta))$ and extend by linearity. ∎

In the following theorem I shall construct an operator $S_\Omega^{(k)} : \bigwedge^1(J^k\pi) \longrightarrow \bigwedge_0^m \pi_{2k-1,k-1} \cap \bigwedge_1^m \pi_{2k-1}$ where again the manifold on which the operator is defined is indicated by a superscript. I shall use the fact that an embedded submanifold always has a tubular neighbourhood; a general proof of this fact may be found in [46].

Theorem 6.3.4 *Suppose given a family of tubular neighbourhoods of $J^{k-r}\pi_1^r$ in $J^{k-r-1}\pi_1^{r+1}$ for $0 \leq r \leq k-2$. Then corresponding to this family there is an $\mathbf{R}$-linear operator $S_\Omega^{(k)} : \bigwedge^1 J^k\pi \longrightarrow \bigwedge_0^m \pi_{2k-1,k-1} \cap \bigwedge_1^m \pi_{2k-1}$ satisfying the conditions*

1. *$(S_\Omega^{(k)}(\sigma))_{j_p^{2k-1}\phi}$ depends only on the germ of σ at $j_p^k\phi$;*
2. *$\pi_{2k,k}^*(\sigma \wedge \Omega) + d_h(S_\Omega^{(k)}(\sigma)) \in \bigwedge_0^{m+1} \pi_{2k,0} \cap \bigwedge_1^{m+1} \pi_{2k}$; and*
3. *$(j^{2k-1}\phi)^*(S_\Omega^{(k)}(\sigma)) = 0$ for every $\phi \in \Gamma_{loc}(\pi)$.*

Proof When $k = 1$ the operator is just S_Ω as previously defined; so suppose $k > 1$. The induction hypothesis is that, for every bundle $\nu : F \longrightarrow M$ and family of tubular neighbourhoods of $J^{k-r-1}\nu_1^r$ in $J^{k-r-2}\nu_1^{r+1}$ where $0 \le r \le k-3$, there is an $\mathbf{R}$-linear operator $S_\Omega^{(k-1)} : \bigwedge^1 J^{k-1}\nu \longrightarrow \bigwedge_0^m \nu_{2k-3,k-2} \cap \bigwedge_1^m \nu_{2k-3}$ such that, for $\tilde{\sigma} \in \bigwedge^1 J^{k-1}\nu$,

$$\nu_{2k-2,k-1}^*(\tilde{\sigma} \wedge \Omega) + d_h(S_\Omega^{(k-1)}(\tilde{\sigma})) \in \textstyle\bigwedge_0^{m+1} \nu_{2k-2,0} \cap \bigwedge_1^{m+1} \nu_{2k-2},$$

$(j^{2k-3}\psi)^*(S_\Omega^{(k-1)}(\tilde{\sigma})) = 0$ for $\psi \in \Gamma_{loc}(\nu)$, and that $(S_\Omega^{(k-1)}(\tilde{\sigma}))_{j_p^{2k-3}\psi}$ depends only on the germ of $\tilde{\sigma}$ at $j_p^{k-1}\psi$.

Choose ν to be $\pi_1 : J^1\pi \longrightarrow M$. Given $\sigma \in \bigwedge^1 J^k\pi$, transfer σ to $\iota_{k-1,1}(J^k\pi)$ along $\iota_{k-1,1}$; extend it to the tubular neighbourhood using the neighbourhood's projection; and then extend it as a smooth 1-form $\tilde{\sigma}$ over the whole of $J^{k-1}\pi_1$. By the induction hypothesis,

$$S_\Omega^{(k-1)}(\tilde{\sigma}) \in \textstyle\bigwedge_0^m(\pi_1)_{2k-3,k-2} \cap \bigwedge_1^m(\pi_1)_{2k-3} \subset \bigwedge^m J^{2k-3}\pi_1$$

and if I write $E^{(k-1)}$ for $(\pi_1)_{2k-2,k-1}^*(\tilde{\sigma} \wedge \Omega) + d_h(S_\Omega^{(k-1)}(\tilde{\sigma}))$ then

$$E^{(k-1)}\tilde{\sigma} \in \textstyle\bigwedge_0^{m+1}(\pi_1)_{2k-2,0} \cap \bigwedge_1^{m+1}(\pi_1)_{2k-2} \subset \bigwedge^{m+1} J^{2k-2}\pi_1.$$

I shall now use the basic relationship $(\pi_l)_{r+s,r} \circ \iota_{r+s,l} = \iota_{r,l} \circ \pi_{l+r+s,l+r}$. This yields $(\pi_1)_{2k-2,0} \circ \iota_{2k-2,1} = \pi_{2k-1,1}$ and $\pi_1 \circ (\pi_1)_{2k-2,0} \circ \iota_{2k-2,1} = \pi_{2k-1}$, and since $\iota_{2k-2,1}^*(E^{(k-1)}\tilde{\sigma}) \in \bigwedge^{m+1} J^{2k-1}\pi$ one obtains

$$\iota_{2k-2,1}^*(E^{(k-1)}\tilde{\sigma}) \in \textstyle\bigwedge_0^{m+1} \pi_{2k-1,1} \cap \bigwedge_1^{m+1} \pi_{2k-1}.$$

By Lemma 6.3.3 I may therefore apply S_Ω to obtain

$$S_\Omega(\iota_{2k-2,1}^*(E^{(k-1)}\tilde{\sigma})) \in \textstyle\bigwedge_0^m \pi_{2k-1,0} \cap \bigwedge_1^m \pi_{2k-1}.$$

Furthermore, from the relation $(\pi_1)_{2k-3,k-2} \circ \iota_{2k-3,1} = \iota_{k-2,1} \circ \pi_{2k-2,k-1}$,

$$\iota_{2k-3,1}^*(S_\Omega^{(k-1)}(\tilde{\sigma})) \in \textstyle\bigwedge_0^m \pi_{2k-2,k-1} \cap \bigwedge_1^m \pi_{2k-2}.$$

Consequently I may define $S_\Omega^{(k)}$ by

$$\begin{aligned} S_\Omega^{(k)}(\sigma) &= \pi_{2k-1,2k-2}^*\iota_{2k-3,1}^*(S_\Omega^{(k-1)}(\tilde{\sigma})) + S_\Omega(\iota_{2k-2,1}^*(E^{(k-1)}\tilde{\sigma})) \\ &\in \textstyle\bigwedge_0^m \pi_{2k-1,k-1} \cap \bigwedge_1^m \pi_{2k-1}. \end{aligned}$$

Finally, using $(\pi_1)_{2k-2,k-1} \circ \iota_{2k-2,1} = \iota_{k-1,1} \circ \pi_{2k-1,k}$ and $\iota_{k-1,1}^*\tilde{\sigma} = \sigma$,

$$\pi_{2k,k}^*(\sigma \wedge \Omega) + d_h(S_\Omega^{(k)}(\sigma)) = (\pi_{2k,2k-1}^* + d_h S_\Omega)(\iota_{2k-2,1}^*(E^{(k-1)}\tilde{\sigma}))$$

which is an element of $\bigwedge_0^{m+1}\pi_{2k,0}\cap\bigwedge_1^{m+1}\pi_{2k}$ by virtue of the properties of S_Ω. Note also that $S_\Omega^{(k)}$ does not depend on the particular extension of σ to the whole of $J^{k-1}\pi_1$ but only on its value in a neighbourhood of $\iota_{k-1,1}(J^k\pi)$ because the operations used in the construction are all local. Finally, if $\phi\in\Gamma_{loc}(\pi)$,

$$\begin{aligned}&(j^{2k-1}\phi)^*(S_\Omega^{(k)}(\sigma))\\ =\;&(j^{2k-3}(j^1\phi))^*S_\Omega^{(k-1)}(\tilde\sigma)+(j^{2k-1}\phi)^*(S_\Omega(\iota_{2k-2,1}^*(E^{(k-1)}\tilde\sigma)))\end{aligned}$$

where the first term vanishes by the induction hypothesis because $j^1\phi$ is a local section of π_1, and the second term vanishes by virtue of the properties of S_Ω. ∎

Corollary 6.3.5 *If $L\in C^\infty(J^k\pi)$ where $1\le k<\infty$ then a Cartan form for L may be constructed globally by*

$$\Theta_L=S_\Omega^{(k)}(dL)+\pi_{2k-1,k}^*L\Omega.$$

∎

The preceding argument provides a satisfactory demonstration of the existence of a suitable Cartan form; however the uniqueness of such a form is a rather more subtle affair. The results may be summarised as follows:

- The Cartan form for first-order Lagrangian theories is unique;
- The Cartan form for Lagrangian theories of arbitrary order over a one-dimensional base manifold is unique;
- When $\dim M\ge 2$, the Cartan form for Lagrangian theories of third order or higher is *not* unique;
- When $\dim M\ge 2$ there are many Cartan forms for second-order Lagrangian theories which satisfy the criteria I have described, but it is possible to specify more restrictive criteria which permit a unique global choice of Cartan form to be made.

In contrast to this, the Euler-Lagrange form which is constructed from the Cartan form by the equation of first variation is *always* unique (so that when two distinct Cartan forms may be found, their difference will necessarily be annihilated by the horizontal differential d_h). In fact, it is *a priori* possible that a Cartan form could be found in $\bigwedge_0^m\pi_{2k-1,k}\cap\bigwedge_1^m\pi_{2k-1}$, so I shall express the following result in slightly more general terms.

Proposition 6.3.6 *If $L \in C^\infty(J^k\pi)$ and if $\Theta_1, \Theta_2 \in \bigwedge_0^m \pi_{2k-1,k} \cap \bigwedge_1^m \pi_{2k-1}$ have the property that both the $(m+1)$-forms $\delta L_1 = \pi_{2k,k}^* dL \wedge \Omega + d_h\Theta_1$ and $\delta L_2 = \pi_{2k,k}^* dL \wedge \Omega + d_h\Theta_2$ are elements of $\bigwedge_0^{m+1} \pi_{2k,0} \cap \bigwedge_1^{m+1} \pi_{2k}$, then $\delta L_1 = \delta L_2$.*

Proof I shall use local coordinates to show that $d_h(\Theta_1 - \Theta_2) = 0$. First, from $\Theta_1 - \Theta_2 \in \bigwedge_0^m \pi_{2k-1,k-1} \cap \bigwedge_1^m \pi_{2k-1}$,

$$\Theta_1 - \Theta_2 = \sigma^i \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \Omega\right)$$

where the 1-forms σ^i are elements of $\bigwedge_0^1 \pi_{2k-1,k-1}$. If the coordinate representation of each σ^i is

$$(\sigma^i)_j dx^j + \sum_{|I|=0}^{k} (\sigma^i)_\alpha^I du_I^\alpha$$

then

$$d_h\sigma^i = \frac{d(\sigma^i)_j}{dx^m} dx^m \wedge dx^j + \sum_{|I|=0}^{k} \left(\frac{d(\sigma^i)_\alpha^I}{dx^m} dx^m \wedge du_I^\alpha + (\sigma^i)_\alpha^I dx^m \wedge du_{I+1_m}^\alpha\right)$$

and so

$$d_h(\Theta_1 - \Theta_2) = -\sum_{|I|=0}^{k} \left(\frac{d(\sigma^i)_\alpha^I}{dx^i} du_I^\alpha + (\sigma^i)_\alpha^I du_{I+1_i}^\alpha\right) \wedge \Omega.$$

Since $d_h(\Theta_1 - \Theta_2) \in \bigwedge_0^{m+1} \pi_{2k,0}$ the only non-zero terms in this expression are those in $du^\alpha \wedge \Omega$ with coefficients $-\frac{d}{dx^i}((\sigma^i)_\alpha)$. From the vanishing of the other terms, one may calculate recursively that these coefficients equal $\sum_{|I|=k}(-1)^{k+1} \frac{d^{|I|+1}}{dx^{I+1_i}}((\sigma^i)_\alpha^I)$. But for each fixed multi-index J with $|J| = k+1$, $\sum_{I+1_i=J}((\sigma^i)_\alpha^I)$ equals the coefficient of $du_J^\alpha \wedge \Omega$, which is zero. The coefficient of $du^\alpha \wedge \Omega$ is then a sum of derivatives of the coefficients of the $du_J^\alpha \wedge \Omega \quad (|J| = k+1)$ and so itself is zero. ∎

To obtain the coordinate representation of the unique Euler-Lagrange form, I shall make a particular choice of tubular neighbourhood which yields a particular choice of Cartan form. While the coordinate description of the Cartan form may only be valid for this coordinate system, Proposition 6.3.6 implies that the representation of δL is valid in an arbitrary coordinate system. So suppose (x^i, u^α) is a coordinate chart on $U \subset E$ and that, for each s with $1 \leq s \leq k$, (x^i, u_I^α) and $(x^i, u_{;J}^\alpha, u_{1_i;J}^\alpha)$ are the corresponding charts on $U^s \subset J^s\pi$ and $U_1^{s-1} \subset J^{s-1}\pi_1$ respectively, where $|I| \le s$, $|J| \le s-1$. Then a projection $\tau_s : U_1^{s-1} \longrightarrow U^s$ may be defined by the rule

$$\begin{aligned} x^i(\tau_s(a)) &= x^i(a); \\ u_I^\alpha(\tau_s(a)) &= \tfrac{1}{2}u_{;I}^\alpha(a) + \tfrac{1}{2}\left(\sum_{i=1}^m \frac{I(i)}{|I|} u_{1_i;I-1_i}^\alpha(a)\right) \qquad |I| \le s-1; \\ u_I^\alpha(\tau_s(a)) &= \sum_{i=1}^m \frac{I(i)}{|I|} u_{1_i;I-1_i}^\alpha(a) \qquad |I| = s. \end{aligned}$$

Each τ_s may be extended to define a tubular neighbourhood of the whole of $J^s\pi$ in $J^{s-1}\pi_1$, and used to construct the corresponding operator $S_\Omega^{(k)}$. Then given a Lagrangian $L \in C^\infty(J^k\pi)$, the coordinate representation of $\Theta_L = S_\Omega^{(k)}(dL) + \pi_{2k-1,k}^*(L\Omega)$ in the neighbourhood U^{2k-1} is

$$\sum_{|I|=0}^{k-1} \sum_{|J|=0}^{k-|I|-1} (-1)^{|J|} \frac{(I+J+1_i)!\,|I|!\,|J|!}{|I+J+1_i|!\,I!\,J!} \times$$

$$\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial u_{I+J+1_i}^\alpha}\right)(du_I^\alpha - u_{I+1_j}^\alpha dx^j) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \Omega\right) + L\Omega.$$

The corresponding Euler-Lagrange form δL is then

$$\sum_{|J|=0}^{k} (-1)^{|J|} \frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial u_J^\alpha}\right) du^\alpha \wedge \Omega.$$

An example:

- Let π be the trivial bundle $(\mathbf{R}^2 \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates $(x, t; u)$, and let $L \in C^\infty(J^2\pi)$ be given by

  $$L = \tfrac{1}{2}u_x u_t + u_x^3 + u_{xx}^2.$$

  Then in this coordinate system the Cartan form of L is given by

  $$\begin{aligned} \Theta_L &= (3u_x^2 + \tfrac{1}{2}u_t - 2u_{xxx})du \wedge dt - \tfrac{1}{2}u_x du \wedge dx \\ &\quad + 2u_{xx}du_x \wedge dt + (2u_x u_{xxx} - u_{xx}^2 - 2u_x^3 - \tfrac{1}{2}u_x u_t)dx \wedge dt \end{aligned}$$

  and the Euler-Lagrange form of L is

  $$\delta L = (u_{xxxx} + 6u_x u_{xx} + u_{xt})du \wedge dx \wedge dt.$$

  The corresponding Euler-Lagrange equation for a local section ϕ of π which is an extremal of L is

  $$\frac{\partial^4\phi}{\partial x^4} + 6\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x\,\partial t} = 0$$

  so that $\frac{\partial\phi}{\partial x}$ satisfies a version of the Kortweg-de Vries equation.

I shall now justify my earlier remarks about the uniqueness of the Cartan form, starting with the first-order case.

Proposition 6.3.7 *If $S : \bigwedge^1 J^1\pi \longrightarrow \bigwedge_0^m \pi_{1,0} \cap \bigwedge_1^m \pi$ satisfies*

$$\pi_{2,1}^*(\sigma \wedge \Omega) + d_h S(\sigma) \in \bigwedge_0^{m+1} \pi_{2,0} \cap \bigwedge_1^{m+1} \pi_2$$

and $(j^1\phi)^*(S(\sigma)) = 0$ for every $\phi \in \Gamma_{loc}(\pi)$, then for each $\sigma \in \bigwedge^1 J^1\pi$,

$$S(\sigma) = S_\Omega \lrcorner\, \sigma.$$

Proof In local coordinates. As the m-form $S_\Omega \lrcorner\, \sigma - S(\sigma) \in \bigwedge_0^m \pi_{1,0} \cap \bigwedge_1^m \pi_1$, it may be written locally as

$$S_\Omega \lrcorner\, \sigma - S(\sigma) = (\sigma^i)_\alpha du^\alpha \wedge \left(\frac{\partial}{\partial x^i} \lrcorner\, \Omega\right) + f\Omega$$

for some functions $(\sigma^i)_\alpha$, f on $J^1\pi$. Now $(j^1\phi)^*(S_\Omega \lrcorner\, \sigma - S(\sigma)) = 0$ for every $\phi \in \Gamma_{loc}(\pi)$ giving the relationship $f = -(\sigma^i)_\alpha u^\alpha_{1_i}$. Furthermore,

$$d_h(S_\Omega \lrcorner\, \sigma - S(\sigma)) = -\frac{d(\sigma^i)_\alpha}{dx^i} du^\alpha \wedge \Omega - (\sigma^i)_\alpha du^\alpha_{1_i} \wedge \Omega$$

and from $d_h(S_\Omega \lrcorner\, \sigma - S(\sigma)) \in \bigwedge_0^{m+1} \pi_{1,0}$ it follows that each $(\sigma^i)_\alpha = 0$. ∎

Corollary 6.3.8 *The Cartan form in first-order theories is unique.*

∎

Proposition 6.3.9 *If the base manifold M is one-dimensional and if $S : \bigwedge^1 J^k\pi \longrightarrow \bigwedge_0^1 \pi_{2k-1,k-1}$ satisfies*

$$\pi_{2k,k}^*(\sigma \wedge dt) + d_h S(\sigma) \in \bigwedge_0^2 \pi_{2k,0} \cap \bigwedge_1^2 \pi_{2k}$$

and $(j^{2k-1}\phi)^*(S(\sigma)) = 0$ for every $\phi \in \Gamma_{loc}(\pi)$, then $S = S_{dt}^{(k)}$.

Proof If $\sigma \in \bigwedge^1 J^k\pi$ then $(j^{2k-1}\phi)^*(S(\sigma)) = (j^{2k-1}\phi)^*(S_{dt}^{(k)}(\sigma)) = 0$, so that $S(\sigma) - S_{dt}^{(k)}(\sigma)$ is a contact form. By Proposition 6.3.6, $d_h(S(\sigma) - S_{dt}^{(k)}(\sigma)) = 0$ so locally $S(\sigma) - S_{dt}^{(k)}(\sigma) = d_h f$ for some function f on $J^{2k-2}\pi$ using local exactness of d_h. But then

$$\begin{aligned}
\pi_{2k,2k-1}^* d_h f &= \pi_{2k,2k-1}^* h \lrcorner\, df \\
&= h \lrcorner\, h \lrcorner\, df \\
&= h \lrcorner\, d_h f \\
&= h(S(\sigma) - S_{dt}^{(k)}(\sigma)) \\
&= 0
\end{aligned}$$

since the horizontal component of any contact form is zero. ∎

Corollary 6.3.10 *The Cartan form $S^{(k)}_{dt}(dL) + L\,dt$ is unique and has coordinate representation*

$$\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}(-1)^j\frac{d^j}{dt^j}\left(\frac{\partial L}{\partial q^\alpha_{(i+j+1)}}\right)(dq^\alpha_{(i)} - q^\alpha_{(i+1)}dt) + L\,dt.$$

∎

However, the following example shows that when $\dim M \geq 2$ and $k \geq 2$ that my construction does not provide a unique Cartan form.

- Let π be the trivial bundle $(\mathbf{R}^2 \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates $(x, y; u)$. Choose a particular coordinate patch U; let S_1 be an operator $S^{(2)}_{dx\wedge dy}$ defined using the projection $\tau_1 : U^1_1 \longrightarrow U^2$ defined earlier, and let S_2 be an operator defined using the alternative projection τ_2 where $u_{xx}(\tau_2(a)) = u_{x;x}(a) + (u_{x;y}(a) - u_{y;x}(a))$, but the other components of τ_1 and τ_2 are equal. Then in this coordinate patch

  $$S_1(du_{xx}) = (du_x - u_{xx}dx) \wedge dy$$

  but

  $$S_2(du_{xx}) = (du_x - u_{xx}dx) \wedge dy - (du_x \wedge dx + du_y \wedge dy)$$

  so that S_1, S_2 both satisfy the conditions I have specified for the operator $S^{(2)}_{dx\wedge dy}$ but $S_1 \neq S_2$. A similar example can obviously be constructed in cases where $k \geq 2$ and $m \geq 2$. Consequently the global Cartan form in higher-order Lagrangian field theories is not unique.

This example uses a second-order operator $S^{(2)}_\Omega$, and one should note that the alternative operator was constructed using a tubular neighbourhood which was not obtained from a coordinate system in the way I prescribed earlier. However, it is in fact the case that—for second-order systems—my prescription *does* yield a unique operator and a unique Cartan form. This fact has been observed before [43,52] and I shall demonstrate that it is indeed the case. To simplify my notation slightly I shall write $u^\alpha_i, u^\alpha_{ij}$ for the coordinate functions which I would previously have denoted by $u^\alpha_{1_i}, u^\alpha_{1_i+1_j}$; I shall also let $n(ij)$ be the number of distinct indices represented by i and j, so that

$$n(ij) = \frac{|1_i + 1_j|!}{(1_i + 1_j)!}.$$

With this notation,

$$\begin{aligned} S_{\Omega}^{(2)}(dL) &= \left(\left(\frac{\partial L}{\partial u_i^\alpha} - \frac{1}{n(ij)} \frac{d}{dx^j} \frac{\partial L}{\partial u_{ij}^\alpha} \right) (du^\alpha - u_k^\alpha dx^k) \right. \\ &\quad \left. + \frac{1}{n(ij)} \frac{\partial L}{\partial u_{ij}^\alpha} (du_j^\alpha - u_{jk}^\alpha dx^k) \right) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \Omega \right) \end{aligned}$$

in the coordinate patch which is used for the initial definition of the tubular neighbourhood.

Theorem 6.3.11 *There is a unique operator $S_{\Omega}^{(2)}$ which satisfies the conditions of Theorem 6.3.4 and which, in each local coordinate patch, may be constructed from the tubular neighbourhood defined by the coordinate system.*

Proof I shall show that the coordinate representation of $S_{\Omega}^{(2)}(dL)$ given above is unaltered by a change to a different coordinate system. First, suppose (v^β, x^i) to be the new coordinate system on E. The terms in the coordinate representation transform as follows:

$$\begin{aligned} du^\alpha - u_k^\alpha dx^k &= \frac{\partial u^\alpha}{\partial v^\beta}(dv^\beta - v_k^\beta dx^k); \\ du_j^\alpha - u_{jk}^\alpha dx^k &= \frac{\partial u^\alpha}{\partial v^\beta}(dv_j^\beta - v_{jk}^\beta dx^k) + \frac{d}{dx^j}\frac{\partial u^\alpha}{\partial v^\beta}(dv^\beta - v_k^\beta dx^k); \\ \frac{\partial}{\partial u_i^\alpha} &= \frac{\partial v^\gamma}{\partial u^\alpha}\frac{\partial}{\partial v_i^\gamma} + \frac{2}{n(ij)}\frac{d}{dx^j}\left(\frac{\partial v^\gamma}{\partial u^\alpha}\right)\frac{\partial}{\partial v_{ij}^\gamma}; \\ \frac{\partial}{\partial u_{ij}^\alpha} &= \frac{\partial v^\gamma}{\partial u^\alpha}\frac{\partial}{\partial v_{ij}^\gamma}. \end{aligned}$$

Invariance of the coordinate representation follows from a straightforward calculation using

$$\frac{d}{dx^j}\left(\frac{\partial v^\gamma}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial v^\beta}\right) = 0.$$

On the other hand, suppose (u^α, y^j) to be the new coordinate system on E. In this case, the calculations seem to be simpler by letting τ be the tubular neigbourhood projection corresponding to the original coordinate system and writing this in the new coordinate system:

$$\begin{aligned} \tau^*(du^\alpha) &= du^\alpha; \\ \tau^*(du_i^\alpha) &= \tfrac{1}{2}(du_{i;}^\alpha + du_{;i}^\alpha); \\ \tau^*(du_{ij}^\alpha) &= \tfrac{1}{2}(du_{i;j}^\alpha + du_{j;i}^\alpha) + \tfrac{1}{2}a_{ij}^m du_{m;}^\alpha + \tfrac{1}{2}b_{ij}^m du_{;m}^\alpha \end{aligned}$$

where

$$\begin{aligned} a^m_{ij} &= \frac{\partial y^p}{\partial x^i}\frac{\partial y^q}{\partial x^j}\frac{\partial^2 x^m}{\partial y^p\,\partial y^q} \\ b^m_{ij} &= \frac{\partial^2 y^p}{\partial x^i\,\partial x^j}\frac{\partial x^m}{\partial y^p}. \end{aligned}$$

An explicit calculation of $S^{(2)}_\Omega$ in the new coordinates shows that again the coordinate representation is unchanged, this time as a consequence of $a^m_{ij} + b^m_{ij} = 0$. Therefore my specification of $S^{(2)}_\Omega$ in local coordinates gives a well-defined operator on the whole of $J^2\pi$. ∎

Corollary 6.3.12 *It is possible to select a unique Cartan form in second-order field theories.*

∎

6.4 Second-order Jet Fields

As I mentioned in the introduction to this Chapter, the traditional calculus of variations makes considerable use of second-order differential equation fields. In the time-dependent case these are vector fields defined on the product manifold $\mathbf{R}\times TF$ ($\cong J^1\pi$ where $\pi = (\mathbf{R}\times F, pr_1, \mathbf{R})$). A time-dependent second-order differential equation field Γ is then required to satisfy two conditions:

1. $\Gamma \lrcorner\, \sigma = 0$ for every $\sigma \in \bigwedge^1_C(\pi)_1$, or equivalently $\Gamma \lrcorner\, S_{dt} = 0$;
2. $d_\Gamma t = 1$.

In coordinates,

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \Gamma^\alpha \frac{\partial}{\partial \dot{q}^\alpha}.$$

Starting with a time-dependent Lagrangian $L \in C^\infty(\mathbf{R}\times TF)$, one now attempts to define the Euler-Lagrange field Γ_L to satisfy

$$\Gamma_L \lrcorner\, d\Theta_L = 0$$

where $d\Theta_L = dd_{S_{dt}}L$. In this section I shall generalise this construction to the case where $\dim M > 1$ using second-order jet fields as defined in Section 4.2; I shall also write Γ^α_{ij} instead of the more cumbersome $\Gamma^\alpha_{1_i+1_j}$ in the coordinate representation of a second-order jet field Γ.

In view of the way that second-order differential equation fields are defined on TF, one might ask whether a similar attempt would be successful on $J^1\pi$.

Lemma 6.4.1 *A second-order jet field Γ on $J^1\pi$ is a section of $(\pi_{1,0})^1 : J^1\pi_1 \longrightarrow J^1\pi$. If* $\dim M = 1$ *then a map $\Gamma : J^1\pi \longrightarrow J^1\pi_1$ which is a section of both $(\pi_1)_{1,0}$ and $(\pi_{1,0})^1$ is a second-order jet field. If* $\dim M \geq 2$ *then this criterion is inadequate.*

Proof If $\mathrm{Im}(\Gamma) \subset \iota_{1,1}(J^2\pi)$ then $u^\alpha_{;1_j} \circ \Gamma = u^\alpha_{1_j;} \circ \Gamma = u^\alpha_{1_j}$ by Lemma 3.3.2, so that $u^\alpha_{1_j} \circ (\pi_{1,0})^1 \circ \Gamma = u^\alpha_{;1_j} \circ \Gamma = u^\alpha_{1_j}$, demonstrating that $(\pi_{1,0})^1 \circ \Gamma = id_{J^1\pi}$.

If $\dim M = 1$ and Γ is a section of both $(\pi_1)_{1,0}$ and $(\pi_{1,0})^1$ then, now using coordinates $(x, u^\alpha_;, u^\alpha_{x;}, u^\alpha_{;x}, u^\alpha_{x;x})$ on $J^1\pi_1$,

$$\begin{aligned} u^\alpha_{;x} \circ \Gamma &= u^\alpha_x \circ (\pi_{1,0})^1 \circ \Gamma \\ &= u^\alpha_x \circ (\pi_1)_{1,0} \circ \Gamma \\ &= u^\alpha_{x;} \circ \Gamma \end{aligned}$$

so that $\mathrm{Im}(\Gamma) \subset \iota_{1,1} J^2\pi$.

Finally let π now be the trivial bundle $(\mathbf{R}^2 \times \mathbf{R}, pr_1, \mathbf{R})$ with coordinates $(x, y; u)$ and let $a \in \mathbf{R}^2 \times \mathbf{R}$. Then the section of $(\pi_1)_{1,0}$ defined by

$$\begin{aligned} u_{;x}(\Gamma(a)) &= u_x(a), \\ u_{;y}(\Gamma(a)) &= u_y(a), \\ u_{x;y}(\Gamma(a)) &= 1, \\ u_{x;x}(\Gamma(a)) = u_{y;y}(\Gamma(a)) = u_{y;x}(\Gamma(a)) &= 0 \end{aligned}$$

is also a section of $(\pi_{1,0})^1$ but is not a second-order jet field. ∎

A general property of second-order jet fields is that they determine partial decompositions of the vertical tangent bundle to $J^1\pi$. This is a generalisation of the result from tangent-bundle geometry, that each second-order differential equation field $\Gamma : TM \longrightarrow TTM$ determines a decomposition of the bundle $\tau_{TM} : TTM \longrightarrow TM$ as a direct sum $V\tau_M \oplus H_\Gamma \longrightarrow TM$. The proof involves taking the Lie derivative by Γ of the vertical endomorphism S and observing that $Q = \frac{1}{2}(I - d_\Gamma S)$ is a projection operator whose kernel is the set of vertical vectors. The image of Q therefore determines a set of horizontal vectors. I shall carry out a similar construction on the first jet manifold, using the Frölicher-Nijenhuis bracket.

Theorem 6.4.2 *Each second-order jet field Γ on $J^1\pi$ determines a decomposition of the bundle $V\pi_1 \longrightarrow J^1\pi$ as a direct sum $V\pi_{1,0} \oplus H_\Gamma \longrightarrow J^1\pi$: that is, every tangent vector to $J^1\pi$ vertical over M is assigned a unique component which is vertical over E.*

Proof I shall consider the vector-valued $(m+1)$-form $[S_\Omega, \tilde{\Gamma}]$. For every 1-form σ on $J^1\pi$, $[S_\Omega, \tilde{\Gamma}] \lrcorner\, \sigma \in \bigwedge_1^{m+1}(\pi_1)$ and if $\sigma \in \bigwedge_0^1(\pi_1)$ then $[S_\Omega, \tilde{\Gamma}] \lrcorner\, \sigma = 0$;

this may be seen from the coordinate representation of $[S_\Omega, \tilde{\Gamma}]$:

$$[S_\Omega, \tilde{\Gamma}] = (du^\alpha_{1_i} \wedge \Omega) \otimes \frac{\partial}{\partial u^\alpha_{1_i}} - (du^\alpha \wedge \Omega) \otimes \left(\frac{\partial}{\partial u^\alpha} + \frac{\partial \Gamma^\beta_{ij}}{\partial u^\alpha_{1_i}} \frac{\partial}{\partial u^\beta_{1_j}} \right).$$

Writing $I \wedge \Omega$ for the vector-valued $(m+1)$-form defined by $I \wedge \Omega \lrcorner \sigma = \sigma \wedge \Omega$ and putting $Q = \frac{1}{2}(I \wedge \Omega - [S_\Omega, \tilde{\Gamma}])$ then again $Q \lrcorner \sigma \in \bigwedge^{m+1}_1(\pi_1)$ and if $\sigma \in \bigwedge^1_0(\pi_1)$ then $Q \lrcorner \sigma = 0$; in fact

$$Q = (du^\alpha \wedge \Omega) \otimes \left(\frac{\partial}{\partial u^\alpha} + \frac{1}{2} \frac{\partial \Gamma^\beta_{ij}}{\partial u^\alpha_{1_i}} \frac{\partial}{\partial u^\beta_{1_j}} \right)$$

I shall now use the canonical isomorphism between $V^*\pi_1$ and $T^*J^1\pi \wedge \pi_1^*\{\Omega\}$: this is the pointwise version of Lemma 1.3.12. The vector-valued $(m+1)$-form Q defines a mapping (also called Q) from $T^*J^1\pi$ to $T^*J^1\pi \wedge \pi_1^*\{\Omega\}$ by the rule $Q(\sigma_p) = Q(\sigma)_p$ where $\sigma_p \in T^*_p J^1\pi$; if it so happens that $\sigma_p \in \pi_1^* T^*M$ then $Q(\sigma_p) = 0$. But each $\theta_p \in T^*J^1\pi \wedge \pi_1^*\{\Omega\}$ has a representative σ_p satisfying $\theta_p = \sigma_p \wedge \Omega_p$, and any two such representatives differ by an element of $\pi_1^* T^*M$. I may therefore define $Q(\theta_p)$ to equal $Q(\sigma_p)$ where σ_p is a representative of θ_p. The resulting endomorphism of $T^*J^1\pi \wedge \pi_1^*\{\Omega\}$ (and hence of $V^*\pi_1$) yields the dual endomorphism of $V\pi_1$ which is a projection operator expressed in coordinates as

$$\xi^\alpha \left.\frac{\partial}{\partial u^\alpha}\right|_p + \eta^\alpha_i \left.\frac{\partial}{\partial u^\alpha_{1_i}}\right|_p \longmapsto \xi^\alpha \left(\left.\frac{\partial}{\partial u^\alpha}\right|_p + \frac{1}{2} \left.\frac{\partial \Gamma^\beta_{ij}}{\partial u^\alpha_{1_i}}\right|_p \left.\frac{\partial}{\partial u^\beta_{1_j}}\right|_p \right)$$

The kernel of this endomorphism is $V\pi_{1,0}$, and defining its image to be H_Γ gives the required decomposition of $V\pi_1$. ∎

I shall now consider the particular case when a second-order jet field Γ is associated with a Lagrangian $L : J^1\pi \longrightarrow \mathbf{R}$. As one might expect, the association involves $d\Theta_L$, where Θ_L is the Cartan form of L; in coordinates

$$\begin{aligned} d\Theta_L &= \left(\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \, \partial u^\alpha_{1_i}} \right) du^\alpha \wedge \Omega \\ &\quad + \left(\frac{\partial^2 L}{\partial u^\beta \partial u^\alpha_{1_i}} du^\beta + \frac{\partial^2 L}{\partial u^\beta_{1_j} \partial u^\alpha_{1_i}} du^\beta_{1_j} \right) \wedge (du^\alpha - u^\alpha_{1_k} dx^k) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \Omega \right). \end{aligned}$$

Theorem 6.4.3 *Let Γ be a second-order jet field; then*

$$i_{\tilde{\Gamma}} d\Theta_L = (m-1) d\Theta_L + \Gamma^* \delta L.$$

If Γ is integrable then the integral sections of Γ are extremals of L if, and only if,

$$i_{\tilde{\Gamma}} d\Theta_L = (m-1) d\Theta_L.$$

Proof From the coordinate representation of $d\Theta_L$ and $\tilde{\Gamma}$,

$$\begin{aligned} i_{\tilde{\Gamma}} d\Theta_L &= (m-1) d\Theta_L \\ &\quad + \left(\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u^\alpha_{1_i}} - \frac{\partial^2 L}{\partial u^\beta \partial u^\alpha_{1_i}} u^\beta_{1_i} - \frac{\partial^2 L}{\partial u^\beta_{1_j} \partial u^\alpha_{1_i}} \Gamma^\beta_{ij} \right) du^\alpha \wedge \Omega \\ &= (m-1) d\Theta_L + \Gamma^* \delta L. \end{aligned}$$

Now suppose that Γ is integrable. Write $\frac{\delta L}{\delta u^\alpha}$ for the coefficient of $du^\alpha \wedge \Omega$ in the coordinate expression for δL. Then if every integral section of Γ is an extremal of L then for each integral section ϕ,

$$(j^1\phi)^* \Gamma^* \frac{\delta L}{\delta u^\alpha} = (j^2\phi)^* \frac{\delta L}{\delta u^\alpha} = 0.$$

But there is an integral section of Γ through each point of $J^1\pi$, so that $\Gamma^* \frac{\delta L}{\delta u^\alpha} = 0$. Conversely if $i_{\tilde{\Gamma}} d\Theta_L = (m-1) d\Theta_L$ then $\Gamma^* \frac{\delta L}{\delta u^\alpha} = 0$, so if ϕ is an integral section of Γ then $(j^2\phi)^* \frac{\delta L}{\delta u^\alpha} = (j^1\phi)^* \Gamma^* \frac{\delta L}{\delta u^\alpha} = 0$. ∎

I shall call an integrable jet field Γ which satisfies these conditions an *Euler-Lagrange field* for L. When $m = 1$ the condition on $\tilde{\Gamma}$ reduces to $i_{\tilde{\Gamma}} d\Theta_L = 0$; writing $\overline{\Gamma}$ for the time-dependent vector field corresponding to $\tilde{\Gamma}$ (so that $\tilde{\Gamma} = dt \otimes \overline{\Gamma}$) the condition becomes $dt \wedge (\overline{\Gamma} \lrcorner d\Theta_L) = 0$, demonstrating the sense in which Theorem 6.4.3 generalises the one-dimensional result.

Chapter 7

Completely Integrable Evolution Equations

In Section 1.2 I defined an evolution equation to be a vertical generalised vector field $X : J^k\pi \longrightarrow V\pi$. In the past twenty years there has been considerable interest in certain evolution equations which have been termed "completely integrable" by analogy with a completely integrable Hamiltonian system of ordinary differential equations, in the sense that each such equation has associated to it an infinite number of conserved quantities which are in involution with respect to a certain Poisson bracket. These equations have a number of remarkable properties, one of which is that they admit soliton solutions. My intention in this chapter is to relate the study of these equations to certain ideas from the theory of jet bundles which have been developed earlier, and so I shall only discuss those aspects of these equations which are relevant to this point of view.

In Section 7.1, I shall briefly describe the Hamiltonian structure of these evolution equations in terms of jet bundles. In Section 7.2, I describe the inverse scattering transform method for solving these equations, interpreting this transform as an example of a parametrised connection in the sense of Section 4.3, and I mention the rôle of the group $SL(2,\mathbf{R})$ in a geometrical description of this transformation. Finally in Section 7.3 I discuss how completely integrable evolution equations may be considered in terms of commuting vector fields on an infinite-dimensional graded Lie algebra, an interpretation which may be found in [28,53]; I also include some original material (submitted for publication in [64]) which explains the construction in terms of this present dissertation, using the infinite tangent manifold of a semisimple Lie group.

7.1 The Variational Structure of Evolution Equations

In this section I shall explain what is meant by saying that a system of evolution equations is a Hamiltonian system. Loosely, one would expect to define the system as Hamiltonian if the "vector field" defining the equations was an element of the image space of some Hamiltonian structure $\sharp$, where I have yet to specify in detail just what kind of entity this structure could be. To motivate this discussion, I shall describe briefly the way in which one may attempt to construct an analogy between the finite-dimensional systems discussed in Section 6.1 and the infinite-dimensional systems represented by evolution equations. My interpretation is similar to that given in [40]; see also [41,55,56] and the references therein.

On a finite-dimensional manifold, each vector field (whether Hamiltonian or not) locally represents a set of ordinary differential equations, and integrating those equations gives the flow of the vector field. Choosing a point in the manifold represents a choice of initial conditions, and the integral curve through that point represents the solution of the equations with the given initial condition. The manifold therefore represents all possible states of the system, and the flow of the vector field shows how the states change as time passes. With an evolution equation, however, the objects of interest which change with time—the fields—are themselves functions of spatial variables and so (if they are going to form a manifold at all) will form an infinite-dimensional manifold. In the context of this dissertation a field is a local section of a bundle (E, π, M) and so one would be interested in the "tangent vectors" at a "point" $\phi \in \Gamma_{loc}(\pi)$ which one could define by considering 1-parameter families of local sections ϕ_t and specifying a suitable equivalence relation on these families. For each $p \in \text{domain}(\phi)$ the family ϕ_t would define a vertical tangent vector $[t \longmapsto \phi_t(p)] \in V_{\phi(p)}\pi$ and equivalent families would define the same vertical tangent vector. A "vector field" would therefore associate with each $\phi \in \Gamma_{loc}(\pi)$ an equivalence class of 1-parameter families ϕ_t, and therefore with each pair (ϕ, p) a vertical tangent vector in $V_{\phi(p)}\pi$.

To study the most general kind of "vector field" in this way would require a consideration of manifolds of maps, a complicated topic which I intend to avoid. (An indication of the complexity involved may be seen in [57].) Indeed for my purposes such complexity is not necessary, for I shall only consider those "vector fields" on $\Gamma_{loc}(\pi)$ where the vertical tangent vector corresponding to the pair (ϕ, p) depends only on the value of ϕ and some finite number of its derivatives at p—that is, on some finite jet $j^k_p\phi$. I shall further assume that the value of k is bounded over all the pairs (ϕ, p). In particular, a "vector field" with this property is *local*: the tangent vector in $V_{\phi(p)}\pi$ does not depend on the behaviour of ϕ at any other specific points in its domain. I may therefore represent such a "vector field" by a map $X : J^k\pi \longrightarrow V\pi$ for some k, where the map has the property that $X(j^k_p\phi) \in V_{\phi(p)}\pi$. Such a map is precisely what I have called a *vertical generalised vector field*, and the "tangent vector" given by evaluating X at $\phi \in \Gamma_{loc}(\pi)$ is just the vector field along ϕ given by $\widetilde{X}\phi = X \circ j^k\phi$; in Section 3.2 I demonstrated how such a map X represented a system of evolution equations. Since any such X gives rise to a map $X \circ \pi_{\infty,k} : J^\infty\pi \longrightarrow V\pi$ it will sometimes be convenient to consider the subspace of $\mathcal{X}^v(\pi_{\infty,0})$ containing all those vertical vector fields along $\pi_{\infty,0}$ with a finite order k, or indeed the subspace of $\mathcal{V}(\pi_\infty)$ containing the infinite prolongations of those vector fields. It would therefore seem reasonable to attempt to define a Hamiltonian structure for evolution equations in such a way that Hamiltonian vector fields were represented in this form rather than as "vector fields" on $\Gamma_{loc}(\pi)$.

The next step in this development is to obtain a suitable method of representing "functions" and "1-forms" on $\Gamma_{loc}(\pi)$, and of defining a suitable "exterior derivative". First I shall consider a suitable definition of "1-forms". Since I have represented "vector fields" as sections of the bundle $\pi_k^*(V\pi) \longrightarrow J^k\pi$ for some k, a suitable interpretation of a "1-form" would be as a section of the dual bundle $\pi_k^*(V^*\pi) \longrightarrow J^k\pi$; I shall call such a section a *vertical generalised 1-form*. Now suppose the base manifold M to be orientable with given volume form Ω and also write Ω for $\pi^*\Omega$; then each vertical cotangent vector $[\eta] \in V_a\pi$ corresponds to a unique element $\eta \wedge \Omega_a \in \bigwedge^{m+1} T_a^*E$ as in Proposition 1.3.12. I may therefore also consider a vertical generalised 1-form as a particular type of $(m+1)$-form on $J^k\pi$, namely an element of the space $\bigwedge_0^{m+1}\pi_{k,0} \cap \bigwedge_1^{m+1}\pi_k$, and it is significant that if $k = 2l$ and $L \in C^\infty(J^l\pi)$ then the Euler-Lagrange form δL is an element of this space. To avoid having to consider particular values of k it is again convenient to use the pull-back of elements of this space to $J^\infty\pi$, and thus to consider instead the subspace of $\bigwedge_0^{m+1}\pi_{\infty,0} \cap \bigwedge_1^{m+1}\pi_\infty$ containing those $(m+1)$-forms pulled back from some finite jet manifold $J^k\pi$ as the space of vertical generalised 1-forms. The *order* of the vertical generalised 1-form is the smallest value of k such that the 1-form has been pulled back from $J^k\pi$.

Now suppose that $X \in \mathcal{X}^v(\pi_{\infty,0})$ and $\sigma \in \bigwedge_0^{m+1}\pi_{\infty,0} \cap \bigwedge_1^{m+1}\pi_\infty$ have order k, l respectively. The contraction $X \lrcorner\, \sigma$ is well-defined as an element of $\bigwedge_0^m \pi_\infty$ and in fact is the pull-back of an element of $\bigwedge_0^m \pi_{\max\{k,l\}}$. Using the pull-back of the volume form Ω to $J^\infty\pi$ one may identify $X \lrcorner\, \sigma$ with a function on $J^\infty\pi$. This then defines a map from $\Gamma_{loc}(\pi)$ to the set of real-valued functions defined locally on M, by the rule $\pi \longmapsto (X \lrcorner\, \sigma)[\phi]$ where $(X \lrcorner\, \sigma)[\phi]\Omega = (X \lrcorner\, \sigma) \circ j^\infty\phi$. I shall call $X \lrcorner\, \sigma$ a *functional*; as usual the *order* of the functional will be the smallest value of k for which it is the pull-back of an element of $\bigwedge_0^m \pi_k$, and of course any element of $\bigwedge_0^m \pi_\infty$ with a finite order will be a functional. The order of $X \lrcorner \sigma$ will then be not greater than the maximum of the individual orders of X and σ, although equality is not guaranteed—for example, if the supports of X and σ do not intersect then the order of $X \lrcorner \sigma$ will be zero.

Next I must consider a suitable "exterior derivative" mapping functionals to vertical generalised 1-forms. So suppose $F\Omega \in \bigwedge_0^m \pi_\infty$ is a functional of order k. The ordinary exterior derivative $d(F\Omega)$, although an element of $\bigwedge_1^{m+1} \pi_\infty$, is in general *not* an element of $\bigwedge_0^{m+1} \pi_{\infty,0}$ (in coordinates it will contain terms in du_I^α where $|I| > 0$). However there *is* a unique element of $\bigwedge_0^{m+1} \pi_{\infty,0} \cap \bigwedge_1^{m+1} \pi_\infty$ associated with $d(F\Omega)$, and that is the Euler-Lagrange form δF. (Although my definition of the Euler-Lagrange form in Chapter 6 considered functions on $J^k\pi$ and $(m+1)$-forms on $J^{2k}\pi$, the whole construction may be pulled back without difficulty to $J^\infty\pi$.) In coordinates,

$$\delta F = \frac{\delta F}{\delta u^\alpha} du^\alpha \wedge \Omega = \sum_{|I|=0}^{2k} (-1)^{|I|} \frac{d^{|I|}}{dx^I} \left(\frac{\partial F}{\partial u_I^\alpha} \right) du^\alpha \wedge \Omega$$

and indeed the variational derivative $\frac{\delta F}{\delta u^\alpha}$ was seen classically as the generalisation to functionals of the ordinary partial derivative of functions.

I now have most of the ingredients needed to define a variational Hamiltonian structure by analogy with $X_H = \sharp dH$ for finite-dimensional systems. However, at this point I wish to recall my remark from Section 6.1 that finite-dimensional Hamiltonians were only determined up to an equivalence relation, and that if the Hamiltonian structure was non-degenerate then two equivalent Hamiltonians differed by an element of the image of the map $\mathbf{R} \longrightarrow \bigwedge^0 M$ in the de Rham complex

$$0 \longrightarrow \mathbf{R} \longrightarrow \bigwedge^0 M \longrightarrow \bigwedge^1 M \longrightarrow \ldots.$$

In the present context, one would expect Hamiltonians to be equivalence classes of functionals in $\bigwedge_0^m \pi_\infty$, where equivalent functionals differed by an element of the image of a suitable differential map $\bigwedge_0^{m-1} \pi_\infty \longrightarrow \bigwedge_0^m \pi_\infty$ from a locally exact complex. In fact the appropriate complex is the *extended variational complex* which is constructed by joining together a complex involving the horizontal differential d_h and a complex involving the operator δ and other higher-order Euler-Lagrange operators:

$$0 \longrightarrow \mathbf{R} \xrightarrow{d_h} \bigwedge_0^1 \pi_\infty \xrightarrow{d_h} \ldots \xrightarrow{d_h} \bigwedge_0^{m-1} \pi_\infty \xrightarrow{d_h}$$

$$\xrightarrow{d_h} \bigwedge_0^m \pi_\infty \xrightarrow{\delta} \bigwedge_0^{m+1} \pi_{\infty,0} \cap \bigwedge_1^{m+1} \pi_\infty \longrightarrow \ldots$$

(The elements of all the spaces above are supposed to have finite order, although I have not indicated this explicitly to avoid the notation becoming even more complex than it is already; I have also used the same symbol δ to represent the mapping from m-forms as I have used to represent the mapping from functions).

Theorem 7.1.1 *The extended variational complex is locally exact at* $\bigwedge_0^m \pi_\infty$.

Proof See [55], Theorem 4.7. ∎

An element of $d_h(\bigwedge_0^{m-1} \pi_\infty)$ is often called a *total divergence*. To see why, let X be a total derivative on $J^\infty\pi$ whose coefficients are of order not greater than k, so that in coordinates

$$X = X^i \frac{d}{dx^i} \quad \text{where} \quad X^i \in \pi^*_{\infty,k}(J^k\pi).$$

Then $X \lrcorner \Omega$ is an element of $\bigwedge_0^{m-1} \pi_\infty$, and indeed every element of $\bigwedge_0^{m-1} \pi_\infty$ is of this form for suitable X. Consequently the coordinate representation of $d_h(X \lrcorner \Omega)$ is

$$d_h\left(X^i \frac{d}{dx^i} \lrcorner \Omega\right) = \frac{dX^i}{dx^i}\Omega = (\mathrm{Div}_\Omega X)\Omega.$$

In fact, $d_h(X \lrcorner \Omega)$ is just $h_X\Omega$ where h_X is the derivation of type h_* corresponding to X; the analogy with the definition of an ordinary divergence as $(\mathrm{div}_\Omega X)\Omega = d_X\Omega$ should be clear.

With this in mind I shall consider a variational Hamiltonian as an equivalence class of functionals, where two functionals are equivalent if their difference is a total divergence. This implies that the integral of a Hamiltonian composed with a jet extension over any compact simply-connected m-dimensional submanifold of M is well-defined, as indeed is the integral over the whole of M if M itself is simply-connected and the integral converges:

$$\int_C (j^k\phi)^*(F\Omega) = \int_C (j^k\phi)^*(F\Omega + (\mathrm{Div}_\Omega X)\Omega).$$

Now at last I can define a Hamiltonian structure which may be applied to the study of evolution equations.

Definition 7.1.2 *A* variational Hamiltonian structure *is an* $\mathbf{R}$*-linear map* $\sharp : \bigwedge_0^{m+1} \pi_{\infty,0} \cap \bigwedge_1^{m+1} \pi_\infty \longrightarrow \mathcal{X}^v(\pi_{\infty,0})$ *which is skew-symmetric and closed:*

1. $\sharp\theta_1 \lrcorner \theta_2 \sim -\sharp\theta_2 \lrcorner \theta_1$;
2. $\sum_{\sigma\in C_3} d_{\sharp\theta_{\sigma(1)}}(\sharp\theta_{\sigma(2)} \lrcorner \theta_{\sigma(3)}) - [\sharp\theta_{\sigma(1)}, \sharp\theta_{\sigma(2)}] \lrcorner \theta_{\sigma(3)} \sim 0$, *using* C_3 *to represent the group of cyclic permutations of* $\{1,2,3\}$.

where $\theta_i \in \bigwedge_0^{m+1}\pi_{\infty,0} \cap \bigwedge_1^{m+1}\pi_\infty$, the bracket is the Frölicher-Nijenhuis bracket of vector fields along $\pi_{\infty,0}$, and $\sim$ indicates equivalence to within a total divergence.

This is very similar to the definition given in Section 6.1 of a (finite-dimensional) Hamiltonian structure. Note however that $\sharp$ need not be linear over functions on $J^\infty\pi$, so that the following definition has a non-trivial content.

Definition 7.1.3 *The* order *of the variational Hamiltonian structure $\sharp$ is*

$$\max\{\operatorname{order}(\sharp\sigma) - \operatorname{order}(\sigma) : \sigma \in \textstyle\bigwedge_0^{m+1}\pi_{\infty,0} \cap \bigwedge_1^{m+1}\pi_\infty, \operatorname{order}(\sigma) < \infty\}.$$

(This is probably not the best definition of order, but it will suffice for my purposes.)

Given a Hamiltonian structure $\sharp$, a variational Hamiltonian system then comprises an equivalence class of functionals $[H\Omega]$ and a vertical generalised vector field $\sharp\delta H$. The corresponding Poisson bracket on functionals is defined by

$$\{F\Omega, G\Omega\} = -d_{\sharp\delta F}(G\Omega).$$

Note that this expression is equivalent to $-\sharp\delta F \lrcorner \delta G$ using the formula for integration by parts, and may therefore be seen to depend only on the equivalence classes of the functionals; the result may also properly be considered as an equivalence class of functionals.

Some examples:

- There is a canonical Hamiltonian structure of order zero on the bundle of vertical cotangent vectors $(V^*\pi, \nu_\pi^*, M)$. For given $p \in M$, the fibre $(\nu_\pi^*)^{-1}(p)$ is canonically diffeomorphic to $T^*(\pi^{-1}(p))$ which carries a symplectic form ω_p. More precisely, if $\eta \in V^*\pi$ with $\tau_E^*(\eta) = a \in E$ and $\nu_\pi^*(\eta) = \pi(a) = p$ then define $(\theta_p)_\eta \in V_\eta^*\nu_\pi^*$ by, for $\xi \in V_\eta\nu_\pi^*$,

  $$(\theta_p)_\eta(\xi) = \eta((\tau_E^*)_*(\xi))$$

  so that θ_p is a 1-form on $(\nu_\pi^*)^{-1}(p)$; ω_p may be defined to equal $d\theta_p$. If $\lambda \in V_\eta^*\nu_\pi^*$ we may define $\sharp\lambda \in V_\eta\nu_\pi^*$ by

  $$\lambda_1(\sharp\lambda_2) = (\omega_p)_\eta(\lambda_1, \lambda_2)$$

  using non-degeneracy of the symplectic form. Thus any vertical generalised 1-form $\sigma : J^\infty\nu_\pi^* \longrightarrow V^*\nu_\pi^*$ of order k gives rise to a vertical generalised vector field $\sharp\sigma : J^\infty\nu_\pi^* \longrightarrow V\nu_\pi^*$ of order k by the rule

  $$(\sharp\sigma)_{j_p^\infty\phi} = \sharp(\sigma_{j_p^\infty\phi}).$$

In coordinates (x^i, u^α) on E and $(x^i, u^\alpha, v_\alpha)$ on $V^*\pi$, a vertical generalised 1-form appears as

$$\sigma = \sigma_\alpha du^\alpha \wedge \Omega + \sigma^\alpha dv_\alpha \wedge \Omega$$

and then

$$\sharp\sigma = \sigma_\alpha \frac{\partial}{\partial v_\alpha} - \sigma^\alpha \frac{\partial}{\partial u^\alpha},$$

and the Poisson bracket of two functionals $F\Omega$, $G\Omega$ is equivalent to

$$\begin{aligned} -\sharp\delta F \,\lrcorner\, \delta G &= -\left(\frac{\delta F}{\delta u^\alpha}\frac{\partial}{\partial v_\alpha} - \frac{\delta F}{\delta v_\alpha}\frac{\partial}{\partial u^\alpha}\right) \\ &\quad \lrcorner \left(\frac{\delta G}{\delta u^\alpha} du^\alpha \wedge \Omega + \frac{\delta G}{\delta v_\alpha} dv^\alpha \wedge \Omega\right) \\ &= \frac{\delta F}{\delta v_\alpha}\frac{\delta G}{\delta u^\alpha} - \frac{\delta F}{\delta u^\alpha}\frac{\delta G}{\delta v_\alpha}. \end{aligned}$$

- Let π now be the trivial bundle $(\mathbf{R} \times \mathbf{R}, pr_1, \mathbf{R})$ with global coordinates (x, u). Then in this coordinate system there is a Hamiltonian structure of order 1 defined as follows: if the vertical generalised 1-form σ is written as

  $$\sigma = f\, du \wedge dx$$

  then $\sharp\sigma$ is the vertical generalised vector field given by

  $$\sharp\sigma = \frac{df}{dx}\frac{\partial}{\partial u}.$$

  One may check that $\sharp$ is skew-symmetric and closed, so is indeed a Hamiltonian structure. Let $H : J^\infty\pi \longrightarrow \mathbf{R}$ be defined by

  $$H = u^3 - \tfrac{1}{2}u_x^2;$$

  then

  $$\sharp\delta H = (6uu_x + u_{xxx})\frac{\partial}{\partial u}$$

  which defines the evolution equation

  $$u_t = u_{xxx} + 6uu_x,$$

  the Korteweg-de Vries equation [30].

- Retaining the same bundle π, now let the second Hamiltonian structure $\tilde{\sharp}$ be defined by

  $$\tilde{\sharp}\sigma = \left(\frac{d^3 f}{dx^3} + 4u\frac{df}{dx} + 2u_x f\right)\frac{\partial}{\partial u}.$$

If $\widetilde{H} : J^\infty\pi \longrightarrow \mathbf{R}$ is defined by $\widetilde{H} = \frac{1}{2}u^2$ then again

$$\widetilde{\sharp}\delta\widetilde{H} = (6uu_x + u_{xxx})\frac{\partial}{\partial u},$$

showing that the Korteweg-de Vries equation is a bi-Hamiltonian system.

There is an interesting and important consequence of the bi-Hamiltonian nature of the Korteweg-de Vries equation, which is connected with a recursion operator. In Section 6.1, I defined the recursion operator R for finite-dimensional bi-Hamiltonian systems to be $\widetilde{\sharp}\circ\flat$ where $\flat$ was the inverse of $\sharp$. A similar definition may be used for variational Hamiltonian systems, and for the Korteweg-de Vries Hamiltonian structures one finds that

$$\widetilde{\sharp}\circ\flat\left(f\frac{\partial}{\partial u}\right) = \left(\frac{d^2 f}{dx^2} + 4uf + 2u_x\int f\right)\frac{\partial}{\partial u}$$

so that in general $\widetilde{\sharp}\circ\flat$ would appear not to be well-defined (after all, $\sharp$ is not injective). However, if

$$K_1 = (6uu_x + u_{xxx})\frac{\partial}{\partial u}$$

is the Korteweg-de Vries field then—since $6uu_x + u_{xxx} = \frac{d}{dx}(3u^2 + u_{xx})$—a calculation shows that

$$K_2 = \widetilde{\sharp}\circ\flat(K_1) = (30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx})\frac{\partial}{\partial u}$$

and one finds that $[K_1, K_2] = 0$. Similarly, the coefficient in K_2 is also a total derivative, so that $\widetilde{\sharp}\circ\flat$ may be applied a second time. In fact, it may be shown that each $K_n = (\widetilde{\sharp}\circ\flat)^{n-1}(K_1)$ is a well-defined evolution field, and that $[K_m, K_n] = 0$. This recursion operator is called the *Lenard operator*, and the evolution equations defined by the evolution fields are called the *higher KdV equations*: see, for example, [55], Theorem 7.24.

7.2 The Inverse Scattering Transform

So far in this dissertation, on the few occasions when I have considered differential equations I have written about definitions and properties. Of course when given a particular differential equation one would also like to find explicit solutions, and the vast range of techniques for use in a variety of circumstances would do credit to Mrs. Beeton. Many of these techniques are actually applications of the theory of Lie symmetry groups, and a comprehensive description of this theory may be found in [55]. In this section I shall instead explain one recently-discovered technique, the inverse scattering transform. This is a method of finding solutions to certain types of non-linear evolution equation such as the Korteweg-de Vries equation mentioned in the previous section (and also to certain other equations such as the sine-Gordon equation which are equivalent to evolution equations although not given in that form). The technique is specially adapted to solving initial-value problems for these equations, and indeed may be considered as an extension to non-linear equations of the method of Fourier transforms for solving linear evolution equations. Although the first part of this section may appear to have very little to do with differential geometry, I shall gradually indicate some of the links which have been discovered.

The original application of the inverse scattering transform was to the Korteweg-de Vries equation itself, and the context was a study of the time-independent one-dimensional Schrödinger equation as an eigenvalue problem [31]. So let $\mathcal{E}$ be the complex vector space of C^∞ functions $f : \mathbf{R} \longrightarrow \mathbf{C}$, and let $L_\phi : \mathcal{E} \longrightarrow \mathcal{E}$ be the Schrödinger operator corresponding to the potential ϕ:

$$L_\phi f(x) = f''(x) - \phi(x) f(x)$$

where $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ is a given C^∞ function (the *potential*) satisfying the asymptotic condition that

$$\lim_{|x| \to \infty} x^2 \phi(x) = 0.$$

A solution of the eigenvalue problem for L_ϕ is then a pair (λ, f) where $\lambda \in \mathbf{C}$, $f \in \mathcal{E}$ and

$$L_\phi f = -\lambda^2 f.$$

(The choice of λ^2 here is conventional; however the square will be seen to arise naturally when I recast this problem in a two-dimensional first-order form.) The *spectrum* of L_ϕ is the set of values $\lambda \in \mathbf{C}$ for which there is a solution (λ, f_λ) of the eigenvalue problem with $f_\lambda \neq 0$. The spectrum always contains the entire non-zero real axis; this is called the *continuous spectrum*. There may also be a finite number of purely imaginary values in the spectrum; these constitute the *discrete spectrum*.

Associated to the spectrum of an operator L_ϕ is a set of quantities called the *scattering data* (the terminology comes from quantum mechanics; I shall not attempt to justify it). This comprises a function $R : \mathbf{R} - \{0\} \longrightarrow \mathbf{C}$ called the *reflection coefficient*; a finite (but possibly empty) set of positive real numbers called, a little confusingly, the *discrete eigenvalues*; and another set of positive real numbers (of the same cardinality as the first set) called the *normalisation coefficients*. These may be defined as follows.

1. For each non-zero real λ, the Jost functions $f_\lambda^+, f_\lambda^- \in \mathcal{E}$ are the solutions of the Volterra integral equations

$$\begin{aligned} f_\lambda^+(x) &= e^{i\lambda x} - \frac{1}{\lambda}\int_x^\infty f_\lambda^+(\xi)\phi(\xi)\sin\lambda(x-\xi)d\xi \\ f_\lambda^-(x) &= e^{-i\lambda x} + \frac{1}{\lambda}\int_{-\infty}^x f_\lambda^-(\xi)\phi(\xi)\sin\lambda(x-\xi)d\xi; \end{aligned}$$

both (λ, f_λ^+) and (λ, f_λ^-) are solutions of the eigenvalue problem for L_ϕ. Since f_λ^+ and f_μ^- both satisfy the eigenvalue equation their Wronskian $W(f_\lambda^+, f_\mu^-)$, defined in general by

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$$

is a fixed complex number independent of x. The reflection coefficient R is then defined by

$$R(\lambda) = \frac{W(f_{(-\lambda)}^+, f_\lambda^-)}{W(f_\lambda^-, f_\lambda^+)}.$$

2. If there are N distinct values of λ^2 corresponding to the discrete spectrum then let $\varsigma_1, \ldots, \varsigma_N \in \mathbf{R}^+$ be such that each $i\varsigma_n$ is contained in the discrete spectrum. Each ς_n is called a discrete eigenvalue.

3. Let $(i\varsigma_n, f_{i\varsigma_n})$ be a solution of the eigenvalue problem for L_ϕ where $f_{i\varsigma_n}$ is normalised so that

$$\int_{-\infty}^\infty (f_{i\varsigma_n}(x))^2 dx = 1.$$

Define the normalisation coefficient ρ_n by

$$\rho_n = \left(\lim_{x\to\infty} e^{\varsigma_n x} f_{i\varsigma_n}(x)\right)^2.$$

The scattering data corresponding to L_ϕ are then the elements of the set

$$S_\phi = \{R, \varsigma_n, \rho_n : 1 \le n \le N\};$$

the reason for considering this set is that the correspondence $\phi \longmapsto S_\phi$ is injective and, indeed, ϕ may be reconstructed explicitly from a knowledge of S_ϕ in the following way.

1. Define the function $M : \mathbf{R} \longrightarrow \mathbf{C}$ by

$$M(x) = \sum_{n=1}^N \rho_n e^{-\varsigma_n x} + \frac{1}{2\pi}\int_{-\infty}^\infty R(\lambda)e^{i\lambda x}d\lambda.$$

2. Define the function $K : \{(x, y) \in \mathbf{R}^2 : x < y\} \longrightarrow \mathbf{C}$ to be the solution of the Fredholm integral equation

$$K(x, y) + M(x + y) + \int_x^\infty K(x, \xi) M(y + \xi) d\xi = 0;$$

this equation is known as the Gel'fand-Levitan-Marchenko equation.

3. Finally, for $x \in \mathbf{R}$ define

$$w(x) = 2 \lim_{y \to x+} K(x, y).$$

One then finds that $\phi = -w'$.

The reconstruction of ϕ from S_ϕ is called the inverse scattering transform.

I shall now explain what all this has to do with the Korteweg-de Vries equation. So suppose that, rather than a single potential function ϕ and set of scattering data S_ϕ, one considers instead 1-parameter families ϕ_t and S_{ϕ_t} such that the map $\phi : (x, t) \longmapsto \phi_t(x)$ is smooth. Given an initial potential function ϕ_0 one may calculate S_{ϕ_0} and then attempt to determine S_{ϕ_t} for $t > 0$; the inverse scattering transform will then enable ϕ_t to be found. It turns out that if one requires ϕ to satisfy one of a certain number of non-linear evolution equations (such as the Korteweg-de Vries equation) then the scattering data evolve in a particularly straightforward way. To see this, consider the operator B defined by

$$Bf(x) = f''(x) + 4\phi_t(x) f(x) - 2\phi_t'(x) \int_x^\infty f(\xi) d\xi$$

(notice that this is just another formulation of the recursion operator obtained in Section 7.1 using the bi-Hamiltonian nature of the Korteweg-de Vries equation). It may be shown that, for an arbitrary polynomial P,

$$R_t(\lambda) = -\frac{1}{4\lambda^2 P(-4\lambda^2)} \int_{-\infty}^\infty (f_{\lambda,t}(\xi))^2 \left(P(B)\phi_t'\right)(\xi) d\xi$$

and that

$$\frac{\partial}{\partial t} R_t(\lambda) = \frac{1}{2i\lambda} \int_{-\infty}^\infty (f_{\lambda,t}(\xi))^2 \frac{\partial}{\partial t} \phi_t(\xi) d\xi$$

where R_t is the 1-parameter family of reflection coefficients corresponding to ϕ_t and $(\lambda, f_{t,\lambda})$ are solutions to the corresponding eigenvalue problems (see, for example, [6]). Then if ϕ satisfies the evolution equation

$$\frac{\partial \phi_t}{\partial t} = P(B)\phi_t'$$

(*a priori* this might be an integro-differential equation owing to the nature of the operator B) then R_t will satisfy the linear differential equation

$$\frac{\partial R_t}{\partial t}(\lambda) = 2i\lambda P(-4\lambda^2)R_t(\lambda)$$

indicating a very simple evolution of R with time. Explicitly,

$$R_t(\lambda) = R_0(\lambda)e^{2i\lambda P(-4\lambda^2)t}.$$

It may also be shown that, in these circumstances, $\varsigma_{n,t} = \varsigma_{n,0}$: that is, the discrete eigenvalues (if any) remain unchanged with time, so that the change in the Schrödinger operator is of the form known as an *isospectral deformation*. Finally, it may be shown that the normalisation coefficients satisfy another linear equation

$$\frac{d}{dt}\rho_{n,t} = 2\varsigma_n P(4\varsigma_n^2)\rho_{n,t}$$

giving, explicitly,

$$\rho_{n,t} = \rho_{n,0}e^{-2\varsigma_n P(4\varsigma_n^2)t}.$$

In particular, one may use this technique when P is the linear polynomial $P(X) = X$, and then

$$\begin{aligned} B\phi_t' &= \phi_t''' + 6\phi_t\phi_t' \\ &= (\phi_t'' + 3\phi_t^2)' \end{aligned}$$

so that ϕ satisfies the Korteweg-de Vries equation

$$\frac{\partial\phi}{\partial t} = \frac{\partial^3\phi}{\partial x^3} + 6\phi\frac{\partial\phi}{\partial x}$$

and when P is the n-th power $P(X) = X^n$ then ϕ will satisfy the n-th higher KdV equation.

Some time after the introduction of the inverse scattering transform for finding solutions to the KdV family, variations on this idea were proposed for solving other evolution equations. In particular, a method of solving the non-linear Schrödinger equation

$$u_t = i(u_{xx} + 2u^2\overline{u})$$

was found in [70]; slightly later a generalisation of this idea was discovered in a form which could be applied to a variety of different equations [1,2]. This latter method is called the AKNS method after the initials of the authors, and I shall explain briefly how the ideas of differential geometry start to appear.

Suppose one considers the eigenvalue problem

$$L_{q,r}v = \lambda v$$

where now $v : \mathbf{R} \longrightarrow \mathbf{C}^2$, $L_{q,r}$ is the matrix differential operator defined by

$$L_{q,r}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} v^{1\prime} - qv^2 \\ rv^1 - v^{2\prime} \end{pmatrix}$$

and $q, r : \mathbf{R} \longrightarrow \mathbf{C}$ are given C^∞ functions (the *potentials*) which satisfy certain asymptotic conditions. By requiring $r(x) = 1$ one obtains the Schrödinger operator L_q and the eigenvalue problem above is (apart from replacing λ by $i\lambda$) just the eigenvalue problem considered earlier for the Korteweg-de Vries equation; the new problem is in this sense a generalisation. In the same way as before one may obtain the scattering data and use the inverse scattering method to solve initial-value problems for one-parameter families v_t. However, with this two-dimensional formulation it is possible to give a different interpretation; I shall for the moment restrict attention to the case when v, q, r and λ are all real.

Write the eigenvalue equation $L_{q,r}v = \lambda v$ explicitly as

$$\begin{pmatrix} v^{1\prime} \\ v^{2\prime} \end{pmatrix} = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}.$$

Now suppose that, instead of two individual potential functions q, r one has two one-parameter families q_t, r_t and that the evolution of the solution v_t takes the linear first-order form

$$\frac{\partial}{\partial t}\begin{pmatrix} (v_t)^1 \\ (v_t)^2 \end{pmatrix} = \begin{pmatrix} A[q_t, r_t] & B[q_t, r_t] \\ C[q_t, r_t] & -C[q_t, r_t] \end{pmatrix}\begin{pmatrix} (v_t)^1 \\ (v_t)^2 \end{pmatrix}$$

where the square brackets indicate dependence on q_t, r_t and their derivatives (so that A, B, C are really functions on some jet manifold) and the eigenvalues λ are independent of time. Solutions v_t will exist only if the Frobenius integrability condition holds, and for particular choices of A, B and C this condition will give equations which q_t, r_t must satisfy. These equations will be just the equations for which the inverse scattering transform provides a solution.

Some examples.

- Suppose $r_t(x) = 1$, $q_t(x) = \phi(x,t)$ and that A, B, C satisfy

$$\begin{aligned} A[q_t] &= 4\lambda^3 + 2\lambda q_t + q_t' \\ B[q_t] &= q_t'' + 2\lambda q_t' + 4\lambda^2 q_t + 2q_t^2 \\ C[q_t] &= -4\lambda^2 - 2q_t. \end{aligned}$$

  Then the Frobenius condition requires ϕ to satisfy the Korteweg-de Vries equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial x^3} + 6\phi \frac{\partial \phi}{\partial x}.$$

- Suppose $q_t(x) = -r_t(x) = \phi(x,t)$ and that $A, B.C$ satisfy

$$\begin{aligned} A[q_t] &= 4\lambda^3 + 2\lambda q_t^2 \\ B[q_t] &= q_t'' + 2\lambda q_t' + 4\lambda^2 q_t + 2q_t^3 \\ C[q_t] &= -q_t'' + 2\lambda q_t' - 4\lambda^2 q_t - 2q_t^3. \end{aligned}$$

  Then the Frobenius condition requires ϕ to satisfy the modified Korteweg-de Vries equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial x^3} + 6\phi^2 \frac{\partial \phi}{\partial x}.$$

- The following example is not in quite the form I have described, but there is sufficient similarity for me to include it here. Suppose

$$q_t(x) = -r_t(x) = -\tfrac{1}{2}\frac{\partial \phi}{\partial x}(x,t)$$

  and that A, B, C satisfy

$$\begin{aligned} A[\phi] &= \frac{1}{4\lambda}\cos\phi \\ B[\phi] = C[\phi] &= \frac{1}{4\lambda}\sin\phi. \end{aligned}$$

  Then the Frobenius condition requires ϕ to satisfy the sine-Gordon equation

$$\frac{\partial^2 \phi}{\partial x\, \partial t} = \sin\phi.$$

I have another reason for wanting to include the sine-Gordon equation at this stage, and that is to recall the links between the AKNS transform, the theory of surfaces of constant negative curvature, and the Lie group $SL(2,\mathbf{R})$. For this part of my discussion I shall adopt the notation $v : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$,

$q, r : \mathbf{R}^2 \longrightarrow \mathbf{R}$ and regard x, t as coordinates on the domain $\mathbf{R}^2$; the space and time equations then become

$$\begin{aligned}
\frac{\partial}{\partial x}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} &= \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \\
\frac{\partial}{\partial t}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} &= \begin{pmatrix} A[q,r] & B[q,r] \\ C[q,r] & -A[q,r] \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}.
\end{aligned}$$

1. Given the functions q, r, A, B and C, each solution v to the above equations determines a metric of constant negative Gaussian curvature on the domain $\mathbf{R}^2$. To see this, rewrite the two equations as a single matrix equation relating differential forms:

$$\begin{pmatrix} dv^1 \\ dv^2 \end{pmatrix} = \begin{pmatrix} \lambda\, dx + A\, dt & q\, dx + B\, dt \\ r\, dx + C\, dt & -(\lambda\, dx + A\, dt) \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

or, more succinctly,

$$dv = \Theta v$$

where Θ is the matrix of 1-forms shown above. The Frobenius condition may be expressed as $d(\Theta v) = 0$, giving

$$d\Theta - \Theta \wedge \Theta = 0.$$

Now define the three 1-forms σ^1, σ^2 and ω by

$$\begin{aligned}
\sigma^1 &= (q+r)dx + (B+C)dt \\
\sigma^2 &= -2(\lambda\, dx + A\, dt) \\
\omega &= (q-r)dx + (B-C)dt
\end{aligned}$$

so that

$$\Theta = \begin{pmatrix} -\frac{1}{2}\sigma^2 & \frac{1}{2}(\omega + \sigma^1) \\ \frac{1}{2}(-\omega + \sigma^1) & \frac{1}{2}\sigma^2 \end{pmatrix}.$$

The Frobenius condition corresponds to the relations

$$\begin{aligned}
d\sigma^1 &= \omega \wedge \sigma^2 \\
d\sigma^2 &= \sigma^1 \wedge \omega \\
d\omega &= \sigma^1 \wedge \sigma^2.
\end{aligned}$$

If a metric g is defined on $\mathbf{R}^2$ by $g = \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2$ then ω is the corresponding connection 1-form and the equation for $d\omega$ indicates that the Gaussian curvature of the metric is -1. [62]

2. It is a classical fact that the sine-Gordon equation is associated with surfaces of constant negative curvature. If S is a surface in $\mathbf{R}^3$ with curvature -1 and the asymptotic lines given by the embedding are used to define a coordinate system (x, y) on S then the metric g may be written as

$$g = (dx)^2 + 2\cos\phi\, dx\, dy + (dy)^2$$

where the juxtaposition of two 1-forms σ^1, σ^2 indicates their symmetric product $\frac{1}{2}(\sigma^1 \otimes \sigma^2 + \sigma^2 \otimes \sigma^1)$ and where ϕ satisfies the sine-Gordon equation

$$\frac{\partial^2 \phi}{\partial x\, \partial y} = \sin\phi.$$

On the other hand, consider the problem of finding maps $v : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ which take their values on the unit sphere $S^2 \subset \mathbf{R}^3$ (a surface of constant *positive* curvature) and which are harmonic maps in the sense described in Section 6.2 with respect to the Minkowski metric $\mathrm{diag}\{+, -\}$ on $\mathbf{R}^2$, and the usual metric on S^2 induced from the Euclidean metric on $\mathbf{R}^3$. (This problem, the "chiral $O(3)$ model for field theories", was studied in [59].) Defining the map $\phi : \mathbf{R}^2 \longrightarrow \mathbf{R}$ by

$$\phi = g\left(v_*\left(\frac{\partial}{\partial x}\right), v_*\left(\frac{\partial}{\partial y}\right)\right)$$

one finds that ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial x\, \partial y} = -\sin\phi.$$

In fact, in a paper which I have written jointly with Mike Crampin the following two results are demonstrated.

Proposition 7.2.1 *If (M, g) is a Riemannian 2-manifold then around each point $p \in M$ there is a coordinate chart in which g may be written*

$$g = \lambda^2 (dx)^2 + 2\cos\phi\, dx\, dy + \lambda^{-2}(dy)^2$$

where $\lambda \in \mathbf{R}^+$ and ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial x\, \partial y} = -K\sin\phi$$

with $K : M \longrightarrow \mathbf{R}$ being the Gaussian curvature of M; such a coordinate chart is called a Tchebychev coordinate system.

Proof See [16], Proposition 2.1. ∎

Proposition 7.2.2 *If a map $q : \mathbf{R}^2 \longrightarrow M$ is represented locally by the identity matrix with respect to null coordinates in the Minkowski metric on $\mathbf{R}^2$ and Tchebychev coordinates on M then q is harmonic. Conversely if $q : \mathbf{R} \longrightarrow M$ is surjective and harmonic then q may be used to define Tchebychev coordinates in a neighbourhood of each point of M (apart possibly on a set of measure zero).*

Proof See [16], Proposition 4.1. ∎

3. One standard model of a complete surface of constant negative curvature is the upper half of the complex plane $H = \{\xi + i\eta : \eta > 0\}$ with metric

$$g = \frac{1}{\eta^2}\left((d\xi)^2 + (d\eta)^2\right).$$

The group $SL(2, \mathbf{R})$ acts on H by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az+b}{cz+d}$$

and each such transformation is an isometry: indeed the isometry group of H is $SL(2, \mathbf{R})/\{\pm 1\}$ and H is diffeomorphic to the coset space $SL(2, \mathbf{R})/SO(2)$. In addition, the two matrix-valued functions used in the AKNS transform

$$\begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix} \text{ and } \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

have zero trace and so may be regarded as taking their values in the Lie algebra $\mathrm{sl}(2, \mathbf{R})$. It may then be shown that there is a map $G : U \longrightarrow SL(2, \mathbf{R})$ where U is an open subset of $\mathbf{R}^2$ such that the matrix of 1-forms Θ satisfies $\Theta = G^{-1}dG$ and such that, if

$$\Theta = \begin{pmatrix} \theta^1 & \theta^2 \\ \theta^3 & -\theta^1 \end{pmatrix}$$

then the 1-forms $G^*(\theta^i)$ satisfy the Maurer-Cartan equations for left-invariant 1-forms on $SL(2, \mathbf{R})$. Furthermore, each such map G determines an explicit local isometry between $\mathbf{R}^2$, with its metric of curvature -1 defined by the AKNS equations, and the upper half-plane H. [10,37]. This latter idea may be extended to a *complex* manifold of constant curvature $SL(2, \mathbf{C})/SO(2, \mathbf{C})$ whose underlying real manifold is diffeomorphic to the tangent manifold TS^2; of course in the complex case one cannot distinguish between positive and negative curvature and therefore another interpretation of this manifold is as the complex 2-sphere $\mathbf{C}S^2$. [36]

4. Yet another interpretation of the AKNS equations is in terms of a connection. When explaining the relationship between jet fields and connections in Section 4.1, I observed that when π was a vector bundle one could define an affine jet field Γ and hence the covariant differential $\nabla\phi$ of a section ϕ of π; in coordinates, one obtained

$$u^\alpha((\nabla\phi)_p(\xi)) = \left(\left.\frac{\partial \phi^\alpha}{\partial x^i}\right|_p - \Gamma^\alpha_{i\beta}(p)\phi^\beta(p) \right) \xi^i$$

where $\xi = \xi^i \left.\frac{\partial}{\partial x^i}\right|_p$. Letting $\Gamma^\alpha_{i\beta}$ $(i = 1, 2)$ be the coefficients of the two matrix functions in the AKNS equations, one sees that these equations may now be written as $\nabla v = 0$, where v is regarded as a section of a certain vector bundle; as the matrix functions take their values in $\mathbf{sl}(2, \mathbf{R})$ this connection may be associated with a connection on a principal fibre bundle with structure group $SL(2, \mathbf{R})$ [14]. In fact, since the connection coefficients $\Gamma^\alpha_{i\beta}$ depend on the potentials q, r and their derivatives it is appropriate to regard this connection as arising from a parametrised jet field as defined in Section 4.3; this jet field is then another example of a Bäcklund map.

This last interpretation is closely linked to a method for constructing suitable inverse scattering equations known as the method of pseudopotentials [25,68]. Given two bundles (E, π, M) and (F, ρ, M) with coordinates (x^i, u^α), (x^i, v^A) and a parametrised jet field $\Gamma : J^k\pi \times_M F \longrightarrow J^1\rho$, the coordinates v^A are termed *pseudopotentials.* Initially one knows a partial differential equation on π; the problem is to find a suitable bundle ρ and a jet field Γ such that the given equation represents the integrability condition for Γ. The technique involves the use of ideals of differential forms, and an explanation of it in the language of jet bundles may be found in [58]. It is important to note that, when the technique is applied to (for example) the Korteweg-de Vries equation, the nature of the bundle ρ is still undetermined: this is because the horizontal vector fields determined by the connection do not form a closed Lie algebra without the imposition of additional constraints which are not inherent in the original formulation of the problem. Early applications of this technique did include such additional constraints. More recently, however, there have been studies of the infinite-dimensional Lie algebra which is generated without these extra constraints, and in the final section of this chapter I shall show how such an infinite-dimensional Lie algebra arises quite naturally: it is just a combination of tangent and cotangent spaces to $T^\infty G$, where G is a semi-simple Lie group (such as $SL(2, \mathbf{R})$).

7.3 $T^\infty G$ and Kac-Moody Algebras

One description of the construction of the AKNS inverse scattering equations is described in [28]. In the notation of that paper (which used complex-valued functions throughout) the x-equation is written $V_x = PV$ and the t-equation $V_t = Q^{(N)}V$. Here, $V : \mathbf{R}^2 \longrightarrow \mathbf{C}^2$, P is the matrix

$$\begin{pmatrix} -i\varsigma & q \\ r & i\varsigma \end{pmatrix},$$

and $Q^{(N)}$ is a polynomial in ς

$$Q^{(N)} = \sum_{j=0}^{N} Q_j \varsigma^{N-j}$$

where each Q_j depends only on q, r and their x-derivatives, and where $Q_j(x,t) \in \mathbf{sl}(2,\mathbf{C})$. For given N one can then find Q_j recursively (apart from constants of integration) by equating coefficients of the powers of ς in the integrability condition $P_t - Q_x^{(N)} + [P, Q^{(N)}] = 0$. Consequently the non-linear evolution equation solved by this version of the AKNS transform is determined by the choice of N. In fact any particular Q_j does not depend on the degree N of the polynomial $Q^{(N)}$ in which it is a coefficient, and this may have suggested the new idea described in [28]. This idea is to consider, not polynomials in ς, but formal infinite series

$$Q = \sum_{j=0}^{\infty} Q_j \varsigma^{-j}$$

where Q now satisfies the equation $Q_x = [P, Q]$: in fact, writing t_1 instead of x, one has $Q_{t_N} = [Q^{(N)}, Q]$ with $Q^{(N)}$ as described above. This description is then placed in a more abstract setting where these equations appear as the flows of a hierarchy of vector fields, and certain theorems applicable to that setting were used to prove commutativity of the vector fields. (For example, by selecting the integrability relations obtained by requiring one fixed vector field to commute with all the others one may obtain a family of non-linear evolution equations such as the higher KdV family.) An exposition of this process, together with much historical background, may be found in [53].

The abstract setting just mentioned is that of the Kac-Moody algebra $\mathbf{sl}(2,\mathbf{C}) \otimes \mathbf{C}[\varsigma, \varsigma^{-1}]$. This is the algebra of formal series

$$X = \sum_{j=-\infty}^{N} X_j \varsigma^j \qquad X_j \in \mathbf{sl}(2,\mathbf{C})$$

with Lie bracket

$$[X,Y] = \sum_{k=-\infty}^{N} \sum_{i+j=k} [X_i, Y_j]\varsigma^k.$$

This algebra is then written as a direct sum $\mathcal{K} \oplus \mathcal{N}$, where

$$\mathcal{K} = \left\{ \sum_{j=-\infty}^{-1} X_j \varsigma^j \right\}, \qquad \mathcal{N} = \left\{ \sum_{j=0}^{N} X_j \varsigma^j \right\}.$$

The bilinear form

$$< X, Y > \ = \sum_{i+j=k} \mathrm{Tr}(X_i Y_j)$$

is used to identify the dual $\mathcal{N}^*$ with

$$\mathcal{K}^{\perp} = \left\{ \sum_{j=-\infty}^{0} X_j \varsigma^j \right\}.$$

As the dual of a Lie algebra, $\mathcal{K}^{\perp}$ is then assigned a Poisson bracket by

$$\{\phi, \psi\}(X) = - < X, [\pi_{\mathcal{N}} \nabla \phi(X), \pi_{\mathcal{N}} \nabla \psi(X)] >$$

for $X \in \mathcal{K}^{\perp}$, where $\pi_{\mathcal{N}}$ is the projection on the subalgebra $\mathcal{N}$ along $\mathcal{K}$, and where a definition using gradients and the bilinear form rather than differentials has been employed. If, for $X \in \mathcal{K} \oplus \mathcal{N}$, $[\nabla \phi(X), X] = 0$ then ϕ is termed *ad-invariant*, and if both ϕ, ψ are ad-invariant then it can be shown that $\{\phi, \psi\} = 0$ so that the corresponding Hamiltonian vector fields commute. By letting S be the map "multiplication by ς", so that

$$S \left(\sum_{j=0}^{N} X_j \varsigma^j \right) = \sum_{j=0}^{N} X_j \varsigma^{j+1}$$

and putting $\phi_k(X) = -\frac{1}{2} < S^k X, X >$ one can easily see that each ϕ_k is ad-invariant so that $\{\phi_k, \phi_m\} = 0$. The flow of the Hamiltonian vector field corresponding to ϕ_k is just the differential equation $Q_{t_k} = [Q^{(k)}, Q]$.

This point of view may be described by saying that there is really no distinguished independent spatial variable x; instead there is an infinite sequence of "times" corresponding to the flows of the vector fields, any one of which may be chosen as the distinguished spatial variable. Similarly there is an infinite number of dependent variables (the coordinates in the Kac-Moody algebra) which are only related by being x-derivatives of one another once a particular x has been chosen. Now on a jet manifold the derivative coordinates may legitimately be regarded as dependent variables in their own right, and with a single independent variable one may restrict attention to the jets of sections at a given point by studying higher-order tangent bundles. It turns out that there is a natural Lie algebra structure on each cotangent space to the infinite tangent manifold of a semi-simple Lie group, and that this is just the upper half of the Kac-Moody algebra denoted earlier by $\mathcal{N}$. Correspondingly there is a natural Poisson bracket on each tangent space to that manifold (denoted earlier by $\mathcal{K}^{\perp}$). The shift operators used to construct the commuting vector fields appear now as canonical isomorphisms between tangent and cotangent spaces to particular *finite* higher-order tangent manifolds to the group, and the Hamiltonians correspond to the Lagrangians of pseudo-Riemannian metrics on those manifolds.

I shall start by describing the Lie algebra structure of $T_a^* T^\infty G$ where G is a semi-simple Lie group and $a \in T_a T^\infty G$; this is induced from the Lie algebra structure on $T^*_{\tau_G^\infty(a)} G$ by repeated application of the total time derivative d_T. In general, of course, taking the total time derivative of an individual cotangent vector makes no sense. However on a Lie group each cotangent vector gives rise to a unique left-invariant 1-form which may therefore be differentiated. The basic structure at the ground-floor level may be summarised as follows.

Lemma 7.3.1 *Let G be a semi-simple Lie group with Lie algebra g and let $h \in G$. Then the Killing form on g defines a canonical vector space isomorphism $\mathrm{g} \cong \mathrm{g}^*$ so that g^* may be given the structure of a Lie algebra. Furthermore, if g^* is regarded as the space of left-invariant 1-forms on G then there is a canonical isomorphism $\mathrm{g}^* \cong T_h^* G$ given by $\omega \longmapsto \omega_h$ so that $T_h^* G$ acquires the structure of a Lie algebra.*

∎

Now suppose that $a \in T^\infty G$ and write a_k for $\tau_G^{\infty,k}(a)$ $(0 \leq k < \infty)$. The vector space $T_a T^\infty G$ has a natural filtration given by

$$\tau_G^{\infty *} T^*_{a_0} G \subset \tau_G^{\infty,1*} T^*_{a_1} TG \subset \ldots \subset \tau_G^{\infty,k*} T^*_{a_k} T^k G \subset \ldots$$

and I shall extend this to a grading by constructing a distinguished complement to $\tau_G^{k,k-1*} T^*_{a_{k-1}} T^{k-1} G$ in $T^*_{a_k} T^k G$.

Proposition 7.3.2 *Let $\eta \in T^*_{a_k} T^k G$. Then η may be written uniquely as $\eta_1 + \eta_2$, where $\eta_1 \in \tau_G^{k,k-1*} T^*_{a_{k-1}} T^{k-1} G$ and $\eta_2 = (d_T^k \omega)_{a_k}$ for some $\omega \in \mathrm{g}^*$.*

Proof The simplest argument counts the dimensions of the spaces involved. If $\dim G = n$ then $\dim T^*_{a_k} T^k G = (k+1)n$ and, since $\tau_G^{k,k-1*}$ is injective, $\dim \tau_G^{k,k-1*} T^*_{a_{k-1}} T^{k-1} G = kn$. Denote the image of g^* under the map $\omega \longmapsto$

$(d_T^k\omega)_{a_k}$ by $\varsigma^k g^*$; if $\omega \in g^*$ is non-zero then it may be written in local coordinates as $\omega_\alpha dq^\alpha$ where at least one $\omega_\alpha(a_0)$ is non-zero. Then $(d_T^k\omega)_{a_k}$ contains in its coordinate expansion a term $(\omega_\alpha dq^\alpha_{(k)})(a_k)$ and so must also be non-zero. The map $g^* \longrightarrow \varsigma^k g^*$ is therefore injective, so that $\dim \varsigma^k g^* = \dim g^* = n$. Furthermore, the same coordinate term shows that $(d_T^k\omega)_{a_k}$ cannot be an element of $\tau_G^{k,k-1*}T^*_{a_{k-1}}T^{k-1}G$, whose intersection with $\varsigma^k g^*$ must therefore be trivial. ∎

By abuse of notation, I shall omit the pull-back maps and write each finite cotangent space $T^*_{a_k}T^kG$ as a direct sum

$$T^*_{a_k}T^kG \cong \varsigma^0 g^* \oplus \varsigma^1 g^* \oplus \ldots \oplus \varsigma^k g^* \cong T^*_{a_{k-1}}T^{k-1}G \oplus \varsigma^k g^*$$

and the infinite cotangent space as the direct sum

$$T^*_aT^\infty G \cong \bigoplus_{k=0}^{\infty} \varsigma^k g^*.$$

(Note that, from my description of infinite jet manifolds in Section 2.2, any cotangent vector in this space must depend on the differentials of only a finite number of derivative coordinates, and therefore may be expressed as the sum of a finite number of terms from the direct sum.) A typical element of $\varsigma^k g^*$ will henceforth be written as $\varsigma^k\eta$, where $\eta \in T^*_{a_0}G$ satisfies $\eta = \omega_{a_0}$, $\varsigma^k\eta = (d_T^k\omega)_{a_k}$ for $\omega \in g^*$. This construction defines an injective linear map $\varsigma : \varsigma^{k-1}g^* \longrightarrow \varsigma^k g^*$ and therefore injective linear maps

$$\begin{aligned} \varsigma : T^*_{a_{k-1}}T^{k-1}G &\longrightarrow T^*_{a_k}T^kG, \\ \varsigma : T^*_aT^\infty G &\longrightarrow T^*_aT^\infty G. \end{aligned}$$

Furthermore, given a point $h \in G$ and local coordinates (q^α) around h, it is always possible to choose a point $a \in T^\infty G$ such that $\tau_G^\infty(a) = h$ and such that $q^\alpha_{(k)}(a) = 0$ for $k \geq 1$; with such a choice of a one finds that

$$\varsigma(dq^\alpha_{(k)}(a)) = dq^\alpha_{k+1}(a).$$

I can now define a Lie bracket operation on $T^*_aT^\infty G$ using a polynomial rule: if $\sum_{i=0}^m \varsigma^i\eta_i$, $\sum_{j=0}^n \varsigma^j\rho_j$ are elements of $T^*_aT^\infty G$ expressed in terms of the grading then their bracket is defined by

$$\left[\sum_{i=0}^m \varsigma^i\eta_i, \sum_{j=0}^n \varsigma^j\rho_j\right] = \sum_{k=0}^{m+n}\sum_{i=0}^{k} \varsigma^k[\eta_i, \rho_{k-i}].$$

The notation is, of course, suggestive of the Kac-Moody algebra described earlier in this section: however, the present construction provides only the upper half of the algebra. The commuting vector fields were defined on the lower half of the Kac-Moody algebra, and this corresponds to the dual $T_aT^\infty G$. In contrast to that approach, I shall not define a Lie algebra structure on the tangent space at this stage, but will be content with constructing a Poisson bracket on it. This must be described using differentials (as in Section 6.1) rather than gradients, as no bilinear form has yet been introduced. So if f, g are smooth functions on $T_aT^\infty G$ then their Poisson bracket $\{f,g\}_a$ is given by

$$\{f,g\}_a(p) = [df_p, dg_p](p) \qquad p \in T_aT^\infty G$$

where df_p, dg_p are elements of $T_p^*T_aT^\infty G$ (which is canonically identified with $T_a^*T^\infty G$). Finally, therefore, a Poisson bracket may be constructed on the whole tangent manifold $TT^\infty G$:

Theorem 7.3.3 *If f, g are smooth functions on $TT^\infty G$ then they have a canonical Poisson bracket given by*

$$\{f,g\}(p) = \{f_a, g_a\}_a(p)$$

where $a \in T^\infty G$, $p \in T_aT^\infty G$ and f_a, g_a are the restrictions of f, g to $T_aT^\infty G$.

∎

Now given the Poisson bracket on $TT^\infty G$, each function on that manifold defines a Hamiltonian vector field. One way of constructing such functions is from fibre-preserving maps $TT^\infty G \longrightarrow T^*T^\infty G$; I shall show how to construct a sequence of such maps which are linear on each fibre. It will not be immediate that the corresponding functions are in involution. However it will be possible to glue together $TT^\infty G$ and $T^*T^\infty G$ so that each fibre has the structure of the complete Kac-Moody algebra studied in [28] and the involution property will then follow from the results about ad-invariant functions quoted earlier.

I shall start with the linear injection $\varsigma : T^*_{a_{k-1}}T^{k-1}G \longrightarrow T^*_{a_k}T^kG$. Its transpose is a map $T_{a_k}T^kG \longrightarrow T_{a_{k-1}}T^{k-1}G$ whose kernel is isomorphic to $\mathfrak{g}$. This new map will also be called ς for reasons to be explained later. The dual to the decomposition of each cotangent space $T^*_{a_k}T^kG$ is then a decomposition of each tangent space as a direct sum of copies of $\mathfrak{g}$; I shall write this as

$$T_{a_k}T^kG \cong \varsigma^{-k}\mathfrak{g} \oplus \ldots \oplus \varsigma^{-1}\mathfrak{g} \oplus \varsigma^0\mathfrak{g} \cong \varsigma^{-k} \oplus T_{a_{k-1}}T^{k-1}G$$

where $\varsigma^0\mathfrak{g} = \ker \varsigma$. With this notation, ς may also be regarded as a map $\varsigma^{-j}\mathfrak{g} \longrightarrow \varsigma^{-j+1}\mathfrak{g}$.

The isomorphism $g \cong g^*$, which will be denoted θ_0, may now be used to construct an isomorphism $T_{a_k}T^kG \cong T^*_{a_k}T^kG$ by letting it act on each copy of g. When $k \geq 1$ there are of course many ways of constructing such an isomorphism; however it is possible to specify a unique isomorphism θ_k by the requirement that it must "commute with ς" in the following sense. If $p \in \varsigma^{-j}g \subset T_{a_k}T^kG$ where $1 \leq j \leq k$ then $\varsigma p \in \varsigma^{-j+1}g$ may be regarded as an element of $T_{a_k}T^kG$ so that $\theta_k\varsigma p \in T^*_{a_k}T^kG$; I shall require $\theta_k\varsigma p$ to equal the element $\varsigma\theta_k p$ specified in a similar way. This requirement may only be satisfied by setting $\theta_k(\varsigma^{-j}g) = \varsigma^{k-j}g^*$ for $0 \leq j \leq k$ which then defines the isomorphism completely. I shall also denote by θ_k the map $TT^kG \longrightarrow T^*T^kG$ which on each fibre is the linear map θ_k just described.

The following result is now straightforward.

Proposition 7.3.4 *There is a canonical family of functions f_k on $TT^\infty G$, quadratic in the fibre variables, such that each f_k is the pull-back $(\tau_G^{\infty,k}{}_*)^*\overline{f}_k$ of a function $\overline{f}_k$ on TT^kG.*

Proof Define $\overline{f}_k : TT^kG \longrightarrow \mathbf{R}$ by $\overline{f}_k(p) = \frac{1}{2}\theta_k(p)(p)$ and $f_k : TT^\infty G \longrightarrow \mathbf{R}$ by $f_k(p) = \overline{f}_k(\tau_G^{\infty,k}{}_*(p))$. ∎

The factor of a half appears in the above definition because the map $(p,q) \longmapsto \theta_k(p)(q)$ actually defines a pseudo-Riemannian metric on T^kG; this map is certainly bilinear, and it is symmetric and non-degenerate as a consequence of similar properties of the Killing form on g. The function $\overline{f}_k$ is then the Lagrangian of this metric.

To see how this construction relates to the full Kac-Moody algebra I shall now endow each tangent space $T_aT^\infty G$ with a Lie bracket by representing each element of the space as an infinite sequence of elements of g (this is not a decomposition as a direct sum of vector spaces since there may be infinitely many non-zero terms). Given an element $p \in T_aT^\infty G$, the k-th component of the corresponding sequence is defined to be the $\varsigma^{-k}g$ component of $\tau_G^{\infty,k}{}_*(p) \in T_{a_k}T^kG$; from the construction of the finite decompositions this will also equal the $\varsigma^{-k}g$ component of each $\tau_G^{\infty,j}{}_*(p)$ whenever $j > k$. I shall adopt the notation

$$p = \sum_{k=0}^{\infty} \varsigma^{-k} p_k$$

where $p_k \in g$, and define a Lie bracket on $T_aT^\infty G$ by the power series multiplication rule. The map ς may also be extended to this space by representing it as a shift operator.

There are now two Lie algebras, $T_a^*T^\infty G$ and $T_aT^\infty G$, representing the upper and lower halves of the Kac-Moody algebra. However, the two halves overlap: they both contain ς^0 terms. When combining them to form the full Lie algebra I shall therefore consider only the subspace of $T_aT^\infty G \oplus T_a^*T^\infty G$ where the two ς^0 components are equal (implicitly using the isomorphism $\theta_0 : \mathrm{g} \longrightarrow \mathrm{g}^*$). The Lie bracket is defined on the combined space by the power series multiplication rule, using the fact that only finitely many terms in the upper half of the algebra are non-zero; this will clearly be consistent with the definitions on the two original spaces. The map ς may also be defined on the complete algebra using the overlap at the ς^0 term, and is an injective shift operator. (This is why the transpose of the original ς on cotangent spaces was also called ς: they are just different parts of the same map.)

With this new interpretation, the functions f_k may be described in the following way. Given a point $p \in T_aT^\infty G$, shift it by ς^k and consider the component of the result in $T_a^*T^\infty G$. This component is just $\tau_G^{\infty,k*}\theta_k\tau_G^{\infty,k}{}_*(p)$, so the value of the function is found by evaluating that component on p. It is not hard to see that (to within sign) the functions f_k restricted to a single fibre $T_aT^\infty G$ are just the Hamiltonians ϕ_k described in the previous work. Since the Poisson brackets have also been constructed in essentially the same way, the following result has been obtained.

Theorem 7.3.5 *The functions f_k are pairwise in involution with respect to the canonical Poisson structure on $TT^\infty G$.*

∎

There is one more feature of the present construction which supports the view that it is a suitable framework for the study of soliton equations. In [53] there is computational evidence that the operator on the Kac-Moody algebra denoted $\varsigma^j\frac{d}{d\varsigma}$ plays an important rôle, and it is instructive to see how this might be represented. In the present interpretation, ς originated with the total time derivative d_T applied to left-invariant differential forms on G. By choosing a suitable point $a \in (\tau_G^\infty)^{-1}(h)$, where $h \in G$, the coordinate representation of ς was simply $dq^\alpha_{(k)}(a) \longmapsto dq^\alpha_{(k+1)}(a)$. In these coordinates, the operator $\frac{d}{d\varsigma}$ would map $dq^\alpha_{(k+1)}(a)$ to $(k+1)dq^\alpha_{(k)}(a)$. If one wrote this operation as the action on cotangent vectors of a type (1,1) tensor, then in these coordinates the tensor would appear as

$$\sum_k (k+1)\frac{\partial}{\partial q^\alpha_{(k+1)}} \otimes dq^\alpha_{(k)}(a).$$

But this is just the coordinate representation of the first vertical endomorphism $S_{dt}^{(\infty)}$ on higher-order tangent bundles; indeed, the decompositions of $T_aT^\infty G$ and $T_a^*T^\infty G$ may equally well be carried out by using $S_{dt}^{(\infty)}$ rather than d_T. In fact $S_{dt}^{(\infty)}$, d_T and the identity operator I together generate a Heisenberg algebra, indicating another link with the theory of soliton equations. Since, as I have shown, the various S operators play such an important part in the calculus of variations, its appearance here may suggest yet more links between these two major theories.

Bibliography

1. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. The inverse scattering transform—Fourier analysis for non-linear problems. *Stud.Appl.Math.*, 53:249–315, 1974.

2. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. Non-linear evolution equations of physical significance. *Phys.Rev.Lett.*, 31:125–127, 1973.

3. R. Abraham and J. E. Marsden. *Foundations of Mechanics.* Reading, Mass: Benjamin/Cummings, 1978.

4. P. Antonini, G. Marmo, and C. Rubano. Alternative Lagrangians and complete integrability: some remarks. *Il Nuovo Cimento*, 86B(1):17–29, 1985.

5. F. Brickell and R. S. Clark. Integrable almost tangent spaces. *J.Diff.Geom.*, 9:557–563, 1974.

6. F. Calogero. Spectral transform and solitons: tools to solve and investigate non-linear evolution equations. 1979. Lecture notes, Institute for Theoretical Physics, Groningen.

7. R. S. Clark and M. R. Bruckheimer. Sur les structures presque tangents. *C.R.Acad.Sci.Paris*, 251:627–629, 1960.

8. M. Crampin. A note on non-Noether constants of the motion. *Phys.Lett.*, 95A:209–212, 1983.

9. M. Crampin. On horizontal distributions on the tangent bundle of a differentiable manifold. *J. London Math.Soc.*, 3:178–182, 1971.

10. M. Crampin. Solitons and $SL(2, \mathbf{R})$. *Phys.Lett.A*, 66(3):170–172, 1978.

11. M. Crampin. Tangent bundle geometry for Lagrangian dynamics. *J.Phys.A.:Math.Gen.*, 16:3755–3772, 1983.

12. M. Crampin and L. A. Ibort. Graded Lie algebras of derivations and Ehresmann connections. *J.Math.Pures et Appl.*, to appear.

13. M. Crampin, G. Marmo, and C. Rubano. Conditions for the complete integrability of a system admitting alternative Lagrangian descriptions. *Phys.Lett.*, 97A(3):88–90, 1983.

14. M. Crampin, F. A. E. Pirani, and D. C. Robinson. The soliton connection. *Lett.Math.Phys.*, 2:15–19, 1977.

15. M. Crampin, W. Sarlet, and F. Cantrijn. Higher order differential equations and higher order Lagrangian mechanics. *Math.Proc.Camb.Phil.Soc.*, 99:565–587, 1986.

16. M. Crampin and D. J. Saunders. The sine-Gordon equation, Tchebychev nets, and harmonic maps. *Rep.Math.Phys.*, to appear.

17. M. de Léon and P. Rodriguez. *Generalised Classical Mechanics and Field Theory.* Amsterdam: North-Holland, 1985.

18. J. Eells and L. Lemaire. A report on harmonic maps. *Bull.London Math.Soc.*, 10:1–68, 1978.

19. C. Ehresmann. Les prolongements d'une variété différentiable: calcul des jets, prolongement principal. *C.R.Acad.Sci.Paris*, 233:598–600, 1951.

20. C. Ehresmann. Les prolongements d'une variété différentiable: covariants différentiels et prolongements d'une structure infinitésimale. *C.R.Acad.Sci.Paris*, 234:1424–1425, 1952.

21. C. Ehresmann. Les prolongements d'une variété différentiable: éléments de contact et élements d'enveloppe. *C.R.Acad.Sci.Paris*, 234:1028–1030, 1952.

22. C. Ehresmann. Les prolongements d'une variété différentiable: l'espace des jets d'ordre r de V_n dans V_m. *C.R.Acad.Sci.Paris*, 233:777–779, 1951.

23. C. Ehresmann. Les prolongements d'une variété différentiable: transitivité des prolongements. *C.R.Acad.Sci.Paris*, 233:1081–1083, 1951.

24. L. P. Eisenhart. *A Treatise on the Differential Geometry of Curves and Surfaces.* New York: Dover, 1960.

25. F. B. Estabrook and H. D. Wahlquist. Prolongation structures, connection theory and Bäcklund transformation. In *Pitman Research Notes 26, Non-Linear Evolution Equations solvable by the Spectral Transform*, pages 64–83, London: Pitman, 1978.

26. C. Ferrario, G. Lo Vecchio, G. Marmo, and G. Morandi. Separability of completely integrable dynamical systems admitting alternative Lagrangian descriptions. *Lett.Math.Phys.*, 9(2):141–148, 1985.

27. C. Ferrario, G. Lo Vecchio, G. Marmo, G. Morandi, and C. Rubano. A separability theorem for dynamical systems admitting alternative Lagrangian descriptions. *Preprint, University of Ferrara*, 1986.

28. H. Flaschka, A. C. Newell, and T. Ratiu. Kac-Moody Lie algebras and soliton equations II. *Physica*, 9D:300–323, 1983.

29. A. Frölicher and A. Nijenhuis. Theory of vector-valued differential forms. *Nederl.Akad. Wetensch.Proc.*, A59:338–359, 1956.

30. C. S. Gardner. The Korteweg-de Vries equation and generalisations IV. *J.Math.Phys.*, 12:1548–1551, 1971.

31. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura. A method for solving the Korteweg-de Vries equation. *Phys.Rev.Lett.*, 19:1095–1097, 1967.

32. H. Goldschmidt and S. Sternberg. The Hamilton-Cartan formalism in the calculus of variations. *Ann.Inst.Fourier*, 23:203–267, 1973.

33. J. Grifone. Structure presque-tangente et connexions I. *Ann.Inst.Fourier*, 22(1):287–334, 1972.

34. J. Grifone. Structure presque-tangente et connexions II. *Ann.Inst.Fourier*, 22(3):291–338, 1972.

35. V. Guillemin and S. Sternberg. *Geometric Asymptotics.* Providence, R.I.: American Mathematical Society, 1977.

36. L. Hodgkin. Complex scattering equations and hyperbolic geometry. 1980. Preprint, Kings College, London.

37. L. Hodgkin. Soliton equations and hyperbolic maps. *Ann.Inst.Henri Poincaré*, 38(1):49–58, 1983.

38. D. W. Kahn. *Introduction to Global Analysis.* New York: Academic Press, 1980.

39. J. Kijowski and W. M. Tulczyjew. *A Symplectic Framework for Field Theories. Lecture Notes in Physics 107*, Berlin: Springer, 1979.

40. Y. Kosmann-Schwarzbach. Hamiltonian systems on fibred manifolds. *Letters in Mathematical Physics*, 5:229–237, 1981.

41. Y. Kosmann-Schwarzbach. On the momentum mapping in field theory. In *Lecture Notes in Mathematics 1139*, pages 25–73, Berlin:Springer, 1985.

42. Y. Kosmann-Schwarzbach. Vector fields and generalised vector fields on fibered manifolds. In *Lecture Notes in Mathematics 792*, pages 307–355, Berlin:Springer, 1980.

43. D. Krupka. Lepagean forms in higher-order variational theory. In *Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics*, pages 197–238, Bologna: Tecnoprint, 1983.

44. B. A. Kuperschmidt. Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalism. In *Lecture Notes in Mathematics 775, Geometric Methods in Mathematical Physics*, pages 162–218, Berlin: Springer, 1980.

45. S. Lang. *Algebra.* Reading, Mass: Addison-Wesley, 1965.

46. S. Lang. *Differential Manifolds.* Reading, Mass: Addison-Wesley, 1972.

47. P. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

48. A. Lichnerowicz. Géométrie et physique. In *Proceedings of the Meeting "Geometry and Physics" October 12–15 1982*, pages 1–9, Bologna: Pitagora Editrice, 1982.

49. M. Lutzky. Isospectral representation of dynamical systems. *Lett.Nuovo Cimento*, 35(12):393–397, 1982.

50. L. Mangiarotti and M. Modugno. Fibred spaces, jet spaces and connections. In *Proceedings of the Meeting "Geometry and Physics" October 12–15 1982*, pages 135–165, Bologna: Pitagora Editrice, 1982.

51. G. Marmo, E. J. Saletan, A. Simoni, and B. Vitale. *Dynamical Systems: A Differential Geometric Approach to Symmetry and Reduction.* Chichester: John Wiley, 1985.

52. J. Muñoz Masqué. An axiomatic characterisation of the Poincaré-Cartan form for second order variational problems. In *Lecture Notes in Mathematics 1139, Differential Geometric Methods in Mathematical Physics*, Berlin: Springer, 1985.

53. A. C. Newell. *Solitons in Mathematics and Physics.* Philadelphia: SIAM, 1985.

54. A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann.of Math.*, 65:391–404, 1957.

55. P. J. Olver. *Applications of Lie Groups to Differential Equations.* Berlin: Springer, 1986.

56. P. J. Olver. The Hamiltonian structure of evolution equations. *Math.Proc.Camb.Phil.Soc.*, 88:71–88, 1980.

57. R. S. Palais. *Foundations of Global Non-linear Analysis.* New York: Benjamin, 1968.

58. F. A. E. Pirani, D. C. Robinson, and W. F. Shadwick. *A Local Jet Bundle Formulation of Bäcklund transformations.* Dordrecht: D. Reidel, 1979.

59. K. Pohlmeyer. Integrable Hamiltonian systems and interactions through quadratic constraints. *Commun.Math.Phys.*, 46:207–221, 1976.

60. J. F. Pommaret. *Systems of Partial Differential Equations and Lie Pseudogroups.* New York: Gordon and Breach, 1978.

61. C. Rogers and W. F. Shadwick. *Bäcklund Transformations and their Applications.* New York: Academic Press, 1982.

62. R. Sasaki. Soliton equations and pseudospherical surfaces. *Nucl.Phys.B*, 154:343–357, 1979.

63. D. J. Saunders. An alternative approach to the Cartan form in Lagrangian field theories. *J.Phys.A:Math.Gen.*, 20:339–349, 1987.

64. D. J. Saunders. Higher-order tangent bundles and the Kac-Moody interpretation of soliton equations. *Phys.Lett.A*, submitted for publication.

65. D. J. Saunders. Jet fields, connections and second-order differential equations. *J.Phys.A:Math.Gen.*, to appear.

66. F. Treves. *Topological Vector Spaces, Distributions and Kernels.* New York: Academic Press, 1967.

67. W. M. Tulczyjew. The Euler-Lagrange resolution. In *Lecture Notes in Mathematics 836, Differential Geometric Methods in Mathematical Physics*, pages 22–48, Berlin: Springer, 1980.

68. H. D. Wahlquist and F. B. Estabrook. Prolongation structures of non-linear evolution equations. *J.Math.Phys.*, 16:1–7, 1975.

69. F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups.* Berlin: Springer, 1983.

70. V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in non-linear media. *Soviet Phys.JETP*, 34:62–69, 1972.